

QUADRATIC DIFFERENTIAL SYSTEMS AND CHAZY EQUATIONS, I

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ABSTRACT. Generalized Darboux–Halphen (gDH) systems, which form a versatile class of three-dimensional homogeneous quadratic differential systems (HQDS's), are introduced. They generalize the Darboux–Halphen (DH) systems considered by other authors, in that any non-DH gDH system is affinely but not projectively covariant. It is shown that the gDH class supports a rich collection of rational solution-preserving maps: morphisms that transform one gDH system to another. The proof relies on a bijection between (i) the solutions with noncoincident components of any ‘proper’ gDH system, and (ii) the solutions of a generalized Schwarzian equation (gSE) associated to it, which generalizes the Schwarzian equation (SE) familiar from the conformal mapping of hyperbolic triangles. The gSE can be integrated parametrically in terms of the solutions of a Papperitz equation, which is a generalized Gauss hypergeometric equation. Ultimately, the rational gDH morphisms come from hypergeometric transformations. A complete classification of proper non-DH gDH systems with the Painlevé property (PP) is also carried out, showing how some are related by rational morphisms. The classification follows from that of non-SE gSE's with the PP, due to Garnier and Carton-LeBrun. As examples, several non-DH gDH systems with the PP are integrated explicitly in terms of elementary and elliptic functions.

1. INTRODUCTION: (G)DH SYSTEMS, CHAZY EQUATIONS

Autonomous differential systems $\dot{x} = Q(x)$, $x = (x_1, \dots, x_d) \in \mathbb{K}^d$, where the base field \mathbb{K} is \mathbb{R} or \mathbb{C} and each component Q_i of $Q: \mathbb{K}^d \rightarrow \mathbb{K}^d$ is a homogeneous quadratic form in x_1, \dots, x_d , are interesting continuous-time dynamical systems that have found application in the physical and biological sciences. The independent variable will be denoted by τ . If an initial condition $x(\tau_0) = x^0 \in \mathbb{K}^d$ is imposed at any point $\tau = \tau_0 \in \mathbb{K}$, an analytic solution $x = x(\tau)$ will exist locally, as a power series in $\tau - \tau_0$. But, global behavior may be difficult to determine.

An algebraic approach to such homogeneous quadratic differential systems (called HQDS's here) is often useful, e.g., when studying the stability of the point $0 \in \mathbb{K}^d$, the possible phase portraits, and system equivalences [39, 41, 50, 66]. To any HQDS $(\mathbb{K}^d, \dot{x} = Q(x))$ there is naturally associated an algebra $\mathfrak{A} = (\mathbb{K}^d, *)$ over \mathbb{K} . Any $x \in \mathfrak{A}$ can be written as $\sum_{i=1}^d x_i e_i$, where e_1, \dots, e_d are basis vectors for \mathbb{K}^d , and the product $*$ on \mathbb{K}^d is defined by $x * x := Q(x)$ and

$$(1.1) \quad x * y := [(x + y) * (x + y) - x * x - y * y] / 2,$$

which is the usual polarization identity. The HQDS then becomes an evolution equation $\dot{x} = x * x$ in \mathfrak{A} . As defined, the product $*$ is commutative but non-associative, meaning not necessarily associative; and there may be no corresponding multiplicative identity element. Any idempotent $p \in \mathfrak{A}$, satisfying $p * p = p$, yields

a 1-parameter family of ray solutions of the HQDS, namely $x(\tau) = -(\tau - \tau_*)^{-1}p$, and any nilpotent $n \in \mathfrak{A}$, satisfying $n * n = 0$, yields a 1-parameter family (i.e., ray) of constant solutions, namely $x(\tau) \equiv Kn$, $K \in \mathbb{K}$.

An analytic map $\Phi: \mathbb{K}^d \rightarrow \mathbb{K}^d$ with $\Phi(0) = 0$ (or more generally, a germ of an analytic function) is called *solution-preserving* from an HQDS $(\mathbb{K}^d, \dot{x} = \tilde{Q}(\tilde{x}))$ to an HQDS $(\mathbb{K}^d, \dot{x} = Q(x))$, if $x(\tau) := \Phi(\tilde{x}(\tau))$ is a local solution of $\dot{x} = Q(x)$ for each local solution $\tilde{x} = \tilde{x}(\tau)$ of $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ with initial condition $\tilde{x}(\tau_0)$ sufficiently near the origin [41, 66]. One writes $Q = \Phi_* \tilde{Q}$. A straightforward case is when Φ is linear and invertible, and hence everywhere defined. If Φ equals some $T \in GL(d, \mathbb{K})$ then $Q = T \circ \tilde{Q} \circ T^{-1}$, and the corresponding algebras $\mathfrak{A}, \tilde{\mathfrak{A}}$ are isomorphic.

The classification of the qualitative behaviors of d -dimensional HQDS's thus reduces to placing commutative, non-associative algebras $\mathfrak{A} = (\mathbb{K}^d, *)$ into equivalence classes of various sorts, often with the aid of algebraic invariants. The space of commutative products $*$ on \mathbb{K}^d has $d^2(d+1)/2$ parameters, of which such invariants are functions. A full classification when $d = 2$ has been carried out [18, 50, 61]; and has revealed, *inter alia*, which real planar HQDS's have unbounded solutions [20]. But the structure of the 18-parameter space of non-associative algebras with $d = 3$, i.e., of quadratic vector fields on \mathbb{R}^3 or \mathbb{C}^3 , is not fully understood.

Any three-dimensional HQDS $\dot{x} = Q(x) =: x * x$, with 18 parameters, can be written component-wise as

$$(1.2) \quad \begin{cases} \dot{x}_1 = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + b_{11}x_2x_3 + b_{12}x_3x_1 + b_{13}x_1x_2, \\ \dot{x}_2 = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + b_{21}x_2x_3 + b_{22}x_3x_1 + b_{23}x_1x_2, \\ \dot{x}_3 = a_{31}x_1^2 + a_{32}x_2^2 + a_{33}x_3^2 + b_{31}x_2x_3 + b_{32}x_3x_1 + b_{33}x_1x_2. \end{cases}$$

It is not known exactly which such systems are integrable, in any sense; though Kovalevskaya exponents [27] are a useful tool. Conditions in terms of them for the existence of polynomial first integrals [62] and for algebraic integrability [67] are known. But only HQDS's of certain types, such as Lotka–Volterra systems, have been subjected to detailed Kovalevskaya–Painlevé analyses [6, 16, 26, 29, 36], or other integrability analyses [25, 46, 52]. Nor have the cases when the individual components x_1, x_2, x_3 of a solution $x = x(\tau)$ satisfy ‘nice’ nonlinear scalar ODE's of low degree been fully characterized. But the piecemeal integration of HQDS's of the form (1.2), with the aid of elliptic, hypergeometric, or other special functions, has been carried on since the 19th-century work of Darboux, Halphen [31], and Brioschi [7]; and Hoyer [36].

A theme of the present paper, and especially of Part II, is the connection between HQDS's on \mathbb{C}^3 and nonlinear scalar ODE's in the *Chazy class*, which are sometimes satisfied by components $x_i = x_i(\tau)$, or their linear combinations. The Chazy class includes all third-order autonomous scalar ODE's of the form

$$(1.3) \quad \ddot{u} = Au\ddot{u} + B\dot{u}^2 + Cu^2\dot{u} + Du^4.$$

(Besides the invariance under translation of τ , note the invariance under $\tau \mapsto \tau/\lambda$ when accompanied by $u \mapsto \lambda u$ or by $(A, B, C, D) \mapsto (\lambda A, \lambda B, \lambda^2 C, \lambda^3 D)$.) Such ODE's have analytic local solutions $u = u(\tau)$, which may not extend analytically or meromorphically to $\mathbb{C} \ni \tau$. ODE's in the Chazy class appear in symmetry reductions of the self-dual Yang–Mills equations for the gauge group $\text{Diff}(S^3)$ [2], and of the Prandtl boundary layer equations [44, 53]; and also in the construction of

$SU(2)$ -invariant hypercomplex manifolds [33]. In many applications what appears is actually a HQDS, from which a Chazy-class ODE can optionally be derived.

The Chazy class was introduced in an early (1911) extension of Painlevé's classification of nonlinear scalar ODE's from the second to the third order. Chazy ([14]; see also [8]) imposed the Painlevé property (PP), according to which no local solution of the ODE may have a branch point; at least, not one that is movable, with a location depending on the choice of initial conditions. Nearly all nonlinear third-order scalar ODE's that have the PP, and lie in the 'polynomial class' (i.e., express \ddot{u} as a polynomial in u, \dot{u}, \ddot{u}), turn out to reduce to the form (1.3) when recessive, sub-dominant terms are omitted. Up to normalization there are 12 possible nonzero choices for the coefficient vector (A, B, C, D) ; equivalently, 12 choices for the element $[A : B : C : D]$ of the weighted projective space $\mathbb{P}^3(1, 1, 2, 3)$. The corresponding ODE's are now labeled Chazy-I through Chazy-XII [17, 59], with Chazy-XI and XII being infinite families, parametrized by a positive integer N .

The best known Chazy equations are Chazy-III and XII. In those two equations, $[A : B : C : D]$ is respectively

$$[2 : -3 : 0 : 0], \quad [2(N^2 - 36) : -3(N^2 + 12) : 48(N^2 - 36) : -4(N^2 - 36)^2].$$

The former, often called the classical Chazy equation or simply *the* Chazy equation, is the $N \rightarrow \infty$ limit of the latter (in which $N \neq 1, 6$ for the PP to obtain). Generically, any solution of Chazy-III or of Chazy-XII with $N \neq 2, 3, 4, 5$, determined by initial data (u, \dot{u}, \ddot{u}) at some point $\tau = \tau_0$, is only locally defined and has a *natural boundary* in the complex τ -plane, beyond which it cannot be analytically or meromorphically continued. The maximal domain of definition of the solution, extending from τ_0 to this movable boundary, is a disk or a half-plane [38].

Chazy himself showed that Chazy-III and XII can be integrated *parametrically* with the aid of a particular second-order *linear* ODE: the Gauss hypergeometric equation (GHE), which is satisfied by the hypergeometric function ${}_2F_1(t)$. The independent variable τ of the Chazy equation is represented as the ratio of a pair of solutions of a certain GHE, and the dependent variable u as the logarithmic derivative of one of the pair. There is an alternative interpretation of solutions of Chazy-III and XII, involving a HQDS. For both ODE's, any local solution $u = u(\tau)$ can be obtained from a local solution $x = x(\tau)$ of a corresponding 3-dimensional HQDS $\dot{x} = Q(x)$ (a permutation-invariant Darboux–Halphen system) as an average of components: $u = (x_1 + x_2 + x_3)/3$. Recently, Chakravarty and Ablowitz [13] produced several additional representations of the solutions of Chazy-III, as combinations of components of the solutions of other such systems.

The focus of Parts I and II of this paper is therefore on a versatile and rather general class of 3-dimensional HQDS's of the form $(\mathbb{C}^3, \dot{x} = Q(x) =: x * x)$, which will be called *generalized Darboux–Halphen* (gDH) *systems*. By definition, any gDH system can be written component-wise as

$$(1.4) \quad \begin{cases} \dot{x}_1 = -a_1(x_1 - x_2)(x_3 - x_1) + (b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2) - c x_2 x_3, \\ \dot{x}_2 = -a_2(x_2 - x_3)(x_1 - x_2) + (b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2) - c x_3 x_1, \\ \dot{x}_3 = -a_3(x_3 - x_1)(x_2 - x_3) + (b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2) - c x_1 x_2, \end{cases}$$

with $(a_1, a_2, a_3; b_1, b_2, b_3; c) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}$. This is the specialization

$$(1.5) \quad a_{ij} = a_i \delta_{ij}, \quad b_{ij} = (2a_i - c) \delta_{ij} - a_i + b_j$$

of the general HQDS (1.2). The system (1.4) will be denoted by $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$. Such systems make up a dimension-7 subspace of the linear space of 3-dimensional HQDS's. Topics to be treated include the integration of gDH systems, by an implicitly algebro-geometric procedure that generalizes the integration of Chazy-III and XII; rational but nonlinear morphisms between gDH systems; the Painlevé properties of gDH systems; the nonlinear ODE's satisfied by (linear combinations of) gDH components x_i ; and the extent to which solutions of the Chazy equations can be represented in this way.

The general linear group $GL(3, \mathbb{C})$ acting on $x \in \mathbb{C}^3$, resp. the special linear group $SL(3, \mathbb{C})$, has 9, resp. 8 parameters. Hence up to linear equivalence, the space of 3-dimensional HQDS's has only $18 - 9 = 9$ parameters; or $18 - 8 = 10$ if equivalence of HQDS's under $x \mapsto \lambda x$, i.e. under $\tau \mapsto \lambda \tau$, is not considered. Also, no nontrivial 1-parameter group of transformations $T \in SL(3, \mathbb{C})$ stabilizes the gDH subspace. It follows that up to linear equivalence, the gDH subspace has a small codimension in the full HQDS parameter space, i.e., $10 - 7 = 3$. That is, it is fairly generic. However, gDH systems are significantly different from the parametrized 3-dimensional HQDS's (of Lotka–Volterra or Hoyer form) that have been integrated or tested for integrability by other authors, as mentioned above.

If a condition $x = x^0 = (x_1^0, x_2^0, x_3^0) \in \mathbb{C}^3$ is imposed at any $\tau_0 \in \mathbb{C}$, an analytic solution $x = x(\tau)$ of any gDH system will exist for τ in a neighborhood of τ_0 . It is easy to see that if two components coincide at τ_0 , they will do so at all τ . (If $x_j = x_k$ then $\dot{x}_j = \dot{x}_k$.) In fact, any gDH system with $c \neq b_1 + b_2 + b_3$ has a 1-parameter family of meromorphic ray solutions, proportional to $(\tau - \tau_*)^{-1}$, i.e.

$$(1.6) \quad x(\tau) = (c - b_1 - b_2 - b_3)^{-1}(\tau - \tau_*)^{-1}(1, 1, 1),$$

in which all components coincide. This family of ‘scale-invariant’ solutions comes from an idempotent of the algebra \mathfrak{A} associated to (1.4): an element $p_0 \propto e_0 := e_1 + e_2 + e_3$, satisfying $p_0 * p_0 = p_0$. If $c = b_1 + b_2 + b_3$ then e_0 is nilpotent (i.e. $e_0 * e_0 = 0$), and any constant function $x(\tau) \equiv x_1^0(1, 1, 1)$ will be a solution.

The $b_1 = b_2 = b_3 =: b$ case of the gDH system (1.4) has been considered by Bureau [9, 10], but its subcase $b = c/2$ is much better known. Any gDH system satisfying $b_1 = b_2 = b_3 = c/2$ will be called a *Darboux–Halphen* (DH) *system* and denoted by $\text{DH}(a_1, a_2, a_3; c)$. Most attention below will be directed to *proper* DH systems, which also satisfy (i) $c \neq 0$ and (ii) $c - a_1 - a_2 - a_3 \neq 0$. By examination, any proper DH system can alternatively be written as

$$(1.7) \quad \begin{cases} \dot{x}_1 = (c/2) [x_1^2 + (1 + \rho\alpha_1)(x_1 - x_2)(x_3 - x_1)] , \\ \dot{x}_2 = (c/2) [x_2^2 + (1 + \rho\alpha_2)(x_2 - x_3)(x_1 - x_2)] , \\ \dot{x}_3 = (c/2) [x_3^2 + (1 + \rho\alpha_3)(x_3 - x_1)(x_2 - x_3)] , \end{cases}$$

$$\rho := 2/(1 - \alpha_1 - \alpha_2 - \alpha_3),$$

where

$$(1.8) \quad \alpha_i = -a_i/(c - a_1 - a_2 - a_3)$$

defines ‘angular’ parameters $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, satisfying $\alpha_1 + \alpha_2 + \alpha_3 \neq 1$.

The system (1.7) will be denoted by $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$. It is a standardized form of the Darboux–Halphen HQDS, which was introduced in 1881 by Halphen [31] and has been studied more recently [2, 32, 49, 55]. For instance, it arises in the conformal mapping of hyperbolic triangles, where the angular parameters have units

of π radians, and $\alpha_1 + \alpha_2 + \alpha_3 \neq 1$ is a non-Euclidean condition. The permutation-invariant DH system used in representing solutions of Chazy-III and XII, mentioned above, turns out to be $\text{DH}(0, 0, 0 | c)$, resp. $\text{DH}(\frac{2}{N}, \frac{2}{N}, \frac{2}{N} | c)$. The less symmetric system $\text{DH}(0, \frac{1}{3}, \frac{1}{2} | c)$, resp. $\text{DH}(\frac{1}{N}, \frac{1}{3}, \frac{1}{2} | c)$, can also be used.

A nice feature of any proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ is that the corresponding non-associative algebra $\mathfrak{A} = (\mathbb{C}^3, *)$ is *unital*: the above idempotent p_0 , which for a proper DH system equals $(2/c)e_0 = (2/c)(e_1 + e_2 + e_3)$, is a multiplicative identity element of \mathfrak{A} . In any proper DH system, $c \neq 0$ can optionally be ‘scaled out’ and omitted from the parameter vector $(\alpha_1, \alpha_2, \alpha_3 | c)$. In many contexts its precise value is unimportant. If unspecified, by default its value will be 2.

It is known that no proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ is algebraically integrable [47, 63]. But each DH system $\text{DH}(a_1, a_2, a_3; c)$, proper or improper, has a Lie point symmetry that the general gDH system does not. It is stable under $\tau \mapsto (A\tau + B)/(C\tau + D)$ for any $\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$, provided that each component x_i is simultaneously transformed in an affine-linear way [32]. This facilitates integration [2, 13, 32]. A related fact is that if $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$ where each N_i is a positive integer or ∞ , with $\alpha_1 + \alpha_2 + \alpha_3 < 1$, the solutions of $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ will have a modular interpretation [32, 49]. If an appropriate initial condition $x(\tau_0) = x^0$ is imposed at any point $\tau = \tau_0$ in the upper half-plane \mathfrak{H} , the components $x_i = x_i(\tau)$ will become *quasi-modular forms*: single-valued functions on \mathfrak{H} that are logarithmic derivatives of conventional modular forms, and transform specially under a triangle subgroup $\Delta(N_1, N_2, N_3)$ of $\text{PSL}(2, \mathbb{R})$. A fundamental domain for this subgroup, acting on \mathfrak{H} , can be chosen to be a hyperbolic triangle with angles $\pi(\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$; and the group elements will reflect this domain and its images through their sides, thereby tessellating \mathfrak{H} . The real axis will be a natural boundary through which $x = x(\tau)$ cannot be continued.

The simplest example is $(N_1, N_2, N_3) = (\infty, 3, 2)$, when the DH system becomes a famous differential system of Ramanujan. The group $\Delta(\infty, 3, 2)$ is the classical modular group $\text{PSL}(2, \mathbb{Z})$. For an appropriate initial condition $x(\tau_0) = x^0$, the components $x_2(\tau), x_3(\tau)$ will be proportional to logarithmic derivatives of the Eisenstein series E_4, E_6 . These are well-known modular forms for $\text{PSL}(2, \mathbb{Z})$, defined on \mathfrak{H} as q -series: power series in the half-plane variable $q = \exp(2\pi i\tau)$. For other (generic) choices of initial condition, the solution $x = x(\tau)$ will come from the preceding by applying an element of $\text{PSL}(2, \mathbb{C})$ to the independent variable τ ; so it will be defined not on \mathfrak{H} , but on some other half-plane or disk.

In general, the 3-dimensionality of the group manifold of $\text{PSL}(2, \mathbb{C})$ corresponds to that of the space of disks (including infinite disks, i.e. half-planes) in the complex τ -plane, to that of the solution space of any system $\text{DH}(\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$, and to that of its space of initial conditions $x(\tau_0)$. The phenomenon of a natural boundary (a ‘wall of poles’) may have been first discovered for Chazy-III and XII, which are associated to $\text{DH}(0, \frac{1}{3}, \frac{1}{2})$ and $\text{DH}(\frac{1}{N}, \frac{1}{3}, \frac{1}{2})$, but in fact it is displayed by any generic solution of $\text{DH}(\alpha_1, \alpha_2, \alpha_3)$ with $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$, provided that $\alpha_1 + \alpha_2 + \alpha_3 < 1$ is satisfied. As will be seen, though, non-DH gDH systems differ significantly from DH systems: their solutions have no natural boundaries.

The chief results of Part I of this paper are the following.

- (1) Any $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$ that is ‘proper’ (see Definition 2.2) can be integrated parametrically with the aid of the Gauss hypergeometric function ${}_2F_1(t)$. The integration procedure, based on Theorem 2.1, extends the known procedure

for closed-form integration of proper DH systems, and hence the integration of the Chazy-III and XII equations in terms of ${}_2F_1(t)$. The accompanying Theorem 2.11, which is a special case, applies the procedure to the integration of any $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$. Both theorems employ not the Gauss hypergeometric equation (GHE), but rather its generalization, the Papperitz equation (PE).

Remarkably, the PE-based integration of proper gDH systems does not require the presence of a 3-parameter group of Lie symmetries. Non-DH gDH systems are covariant under affine transformations $\tau \mapsto A\tau + B$, but not under projective transformations $\tau \mapsto (A\tau + B)/(C\tau + D)$.

(2) For any choice of distinct singular points $t_1, t_2, t_3 \in \mathbb{P}_t^1$, there is a correspondence between the solutions $x = x(\tau)$ of any proper gDH system and the solutions $t = t(\tau)$ of an associated nonlinear third-order ODE: a generalized Schwarzian equation $\text{gS}_{t_1, t_2, t_3}(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$. (See Theorem 2.8; the parameters of the gSE are birationally related to $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ and constitute an alternative parameter vector.) The correspondence is not quite a bijection: it excludes solutions $x = x(\tau)$ with coincident components, such as the ray solution (1.6). If $(t_1, t_2, t_3) = (0, 1, \infty)$, the correspondence is $t = -(x_2 - x_3)/(x_1 - x_2)$.

The $b_1 = b_2 = b_3$ case of this proper gDH \leftrightarrow gSE correspondence was previously found by Bureau [9]. In the DH case the gSE specializes to Eq. (2.47), a Schwarzian ODE (based on a Schwarzian derivative) which is familiar from conformal mapping [54]. That proper DH systems can be integrated with the aid of Schwarzian equations is well known, but the extension to gDH systems and gSE's, the latter having no direct conformal mapping interpretation, is new.

(3) There is a large collection of nonlinear but *rational* solution-preserving maps $x = \Phi(\tilde{x})$ between gDH systems. This is perhaps the most important result of Part I of this paper. Each of these can be viewed as a map of projective planes, i.e., as a map $\Phi: \mathbb{P}_{\tilde{x}}^2 \rightarrow \mathbb{P}_x^2$. If any such map is applied to a HQDS $\dot{\tilde{x}} = \tilde{Q}(\tilde{x}) =: \tilde{x} * \tilde{x}$ of the gDH form that satisfies certain conditions on its parameters, another HQDS $\dot{x} = Q(x) =: x * x$ of the gDH form will result. That is, $Q = \Phi_* \tilde{Q}$. This is the content of Theorem 3.1, accompanied by Tables 3 and 4, and Figure 1.

The existence of these rational morphisms follows from the PE-based integration scheme for proper gDH systems. At base, each comes from a lifting of a PE (on a Riemann sphere \mathbb{P}_t^1) to another (on a Riemann sphere $\mathbb{P}_{\tilde{t}}^1$). The lifting is along a rational map $t = R(\tilde{t})$. Equivalently, each comes from the lifting of a GHE to a GHE, or a gSE to another gSE. In fact, each is associated to a *hypergeometric transformation* (a transformation of a ${}_2F_1(t)$ to a ${}_2F_1(\tilde{t})$), since such transformations come from such liftings of GHE's. The morphisms are denoted by **2**, **3**, etc., in reference to the associated ${}_2F_1$ transformations (quadratic, cubic, etc.), most of which are classical, having been worked out by Goursat [28].

Several of the rational morphisms were found by Harnad and McKay [32], who applied them to certain DH systems with no free parameters, the solutions $x_i = x_i(\tau)$ of which are quasi-modular forms for subgroups of $PSL(2, \mathbb{Z})$. It is now clear that they extend to more general DH systems, and from DH to gDH systems. They can even be applied to some non-gDH HQDS's. For example, the complex morphism denoted by **3_c** yields a solution-preserving map from any 3-dimensional HQDS with the symmetry $\tilde{x}_1 \rightarrow \tilde{x}_2 \rightarrow \tilde{x}_3 \rightarrow \tilde{x}_1$, such as the Leonard–May model of cyclic competition among three species, to another 3-dimensional HQDS.

(4) There is a complete classification of proper non-DH gDH systems with the Painlevé property (PP), given in Theorem 4.1 and Table 6. It is a corollary of the classification of non-Schwarzian gSE's with the PP, begun by Garnier [24] and completed by Carton-LeBrun [12]. The new classification supplements the known result that $\text{DH}(\alpha_1, \alpha_2, \alpha_3)$ has the PP if and only if $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$, where each N_i is a nonzero integer or ∞ . Interestingly, many of the proper non-DH gDH systems with the PP are related by the rational solution-preserving maps $x = \Phi(\tilde{x})$ of § 3. The maps are indicated in the final column of Table 6.

For most proper non-DH gDH systems with the PP, generic solutions $x = x(\tau)$ are doubly periodic, i.e. elliptic. (See Theorem 4.3 and Table 7.) Simple periodicity also occurs, and there are proper gDH systems with the PP for which the poles of $x = x(\tau)$ form a polynomially or exponentially stretched lattice, rather than a regular one. All this is of interest because an irregular pattern of singularities in the complex τ -plane would tend to indicate non-integrability [5]. But no proper non-DH gDH system with the PP has any solution with a natural boundary, unlike the proper DH systems with the PP.

As examples, several proper non-DH gDH systems with the PP, which are ‘basic’ in the sense that they are not images of other such systems under rational morphisms, are integrated in terms of elementary and elliptic functions. Papperitz-based integration suffices, and is facilitated in most cases by the PE having trivial monodromy. (See Examples 4.4–4.7; Example 4.5 includes a complete integration of the $N = 4$ case of Chazy-XI [without recessive terms].) For each of these gDH systems a pair I_1, I_2 of first integrals is derived, which are typically rational in x_1, x_2, x_3 . Some final remarks are made on gDH systems that may lack the PP but are nonetheless integrable. This includes certain improper gDH systems.

Part II of this paper will explore the scalar ODE's satisfied by (linear combinations of) components $x_i = x_i(\tau)$ of the solutions of gDH systems. It will be shown that the abovementioned DH representations of the solutions of the Chazy-III equation [13] are related by the rational morphisms of § 3, and that they can be generalized to DH representations of the solutions of Chazy-XII, and to gDH representations of the solutions of Chazy-I, II, VII, and XI. Solutions of DH systems computed with the aid of the theory of modular forms will also be discussed.

2. HYPERGEOMETRIC INTEGRATION OF GDH SYSTEMS

In this section we examine how the generalized Darboux–Halphen system of (1.4), $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$ can be parametrically integrated with the aid of the Gauss hypergeometric function ${}_2F_1(t)$. We do this ‘backwards’: we start with the Papperitz equation, the most general second-order linear Fuchsian ODE with three singular points on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It generalizes the Gauss hypergeometric equation (GHE). In Theorem 2.1 we derive from any nonzero solution of the Papperitz equation a HQDS satisfied by a triple $(x_1, x_2, x_3) =: x$ of logarithmic derivatives, viewed as a function of τ ; both x and τ are constructed as local functions of $t \in \mathbb{P}^1$. The HQDS turns out to be a (proper) gDH system. This approach is essentially that of Ohyama [55], but basing our treatment on the Papperitz equation rather than on the GHE allows us to treat non-DH gDH systems, quite elegantly. We show in Theorem 2.11 how, *a fortiori*, the Papperitz equation can be used to integrate any proper DH system, and also explain why any DH

system has an extra Lie point symmetry: it is covariant under maps $\tau \mapsto (A\tau + B)/(C\tau + D)$ that are not affine.

Theorem 2.9 states that generic (‘noncoincident’) solutions $x = x(\tau)$ of any proper gDH system are bijective with the solutions $t = t(\tau)$ of an associated nonlinear third-order differential equation, the so-called gSE. (If the singular values on \mathbb{P}_t^1 are chosen to be $0, 1, \infty$, the bijection will be $t = -(x_2 - x_3)/(x_1 - x_2)$.) For any proper DH system the gSE reduces to the Schwarzian equation (2.47), which is well known in the theory of conformal mapping [54]. But the extension to non-DH gDH systems, and in fact the appearance of the gSE in this context, are new.

2.1. Papperitz equations. A Papperitz equation (PE) on \mathbb{P}_t^1 is of the form $\mathcal{L}f = 0$ with $\mathcal{L} := D_t^2 + P_1(t)D_t + P_2(t)$, and is determined by its Riemann P-symbol

$$(2.1a) \quad \left\{ \begin{array}{ccc} t_1 & t_2 & t_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu'_1 & \mu'_2 & \mu'_3 \end{array} \right\},$$

which tabulates the unordered pair of characteristic exponents at each singular point. Here, the singular points are the distinct points $t_1, t_2, t_3 \in \mathbb{P}_t^1$. The respective exponents $\mu_i, \mu'_i \in \mathbb{C}$ must satisfy Fuchs’s relation $\sum_{i=1}^3 (\mu_i + \mu'_i) = 1$ for the equation to be Fuchsian, i.e. for each singular point to be regular, but otherwise they are unconstrained. If no t_i is ∞ , the PE is

$$(2.1b) \quad \left\{ D_t^2 + \left(\sum_{i=1}^3 \frac{1 - \mu_i - \mu'_i}{t - t_i} \right) D_t + \left[\sum_{i=1}^3 \frac{\mu_i \mu'_i (t_i - t_j)(t_i - t_k)}{(t - t_i)^2 (t - t_j)(t - t_k)} \right] \right\} f = 0,$$

the subscripts j, k in the second summand being the two elements of $\{1, 2, 3\}$ other than i , in either order. If one of t_1, t_2, t_3 is ∞ then an obvious limit of (2.1b) is taken. The exponent *differences* $(\alpha_1, \alpha_2, \alpha_3) = (\mu'_1 - \mu_1, \mu'_2 - \mu_2, \mu'_3 - \mu_3)$, defined only up to sign, will play an important role below.

Three special cases (normal forms) of the PE are important in applications and calculations. In each, $(t_1, t_2, t_3) = (0, 1, \infty)$. In the first, with $\mu_1 = \mu_2 = 0$, the P-symbol (2.1a) reduces to

$$(2.2a) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & (1 - \alpha_1 - \alpha_2 - \alpha_3)/2 \\ \alpha_1 & \alpha_2 & (1 - \alpha_1 - \alpha_2 + \alpha_3)/2 \end{array} \right\},$$

and the equation (2.1b) to the GHE

$$(2.2b) \quad \left[D_t^2 + \left(\frac{1 - \alpha_1}{t} + \frac{1 - \alpha_2}{t - 1} \right) D_t + \frac{(1 - \alpha_1 - \alpha_2)^2 - \alpha_3^2}{4t(t - 1)} \right] f = 0.$$

A local solution f of the GHE at $t = 0$ associated to the exponent 0 is the Gauss hypergeometric function

$$(2.3) \quad {}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!},$$

in which $(a)_n := (a)(a+1) \cdots (a+n-1)$ and the hypergeometric parameters $a, b; c$ come from the exponent differences $(\alpha_1, \alpha_2, \alpha_3)$ by

$$(2.4) \quad (a, b; c) = \left(\frac{1}{2}(1 - \alpha_1 - \alpha_2 - \alpha_3), \frac{1}{2}(1 - \alpha_1 - \alpha_2 + \alpha_3); 1 - \alpha_1 \right).$$

Generically ${}_2F_1(a, b; c; t)$ is well-defined: it is defined if c is not a nonpositive integer, i.e., if the exponent α_1 of the GHE at $t = 0$ is not a positive integer. Also generically, a second solution f of the GHE (2.2b) at $t = 0$ is given by

$$(2.5) \quad t^{1-c} {}_2F_1(a - c + 1; b - c + 1; 2 - c; t).$$

It comes from the exponent $\alpha_1 = 1 - c$ at $t = 0$. The local function ${}_2F_1$ can also be used to construct a pair of local solutions to (2.2b) at $t = 1$ and at $t = \infty$.

In the second special case of the PE, with $\mu_1 + \mu'_1 = \mu_2 + \mu'_2 = 1$, the P-symbol (2.1a) reduces to

$$(2.6a) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \frac{(1 - \alpha_1)/2}{(1 + \alpha_1)/2} & \frac{(1 - \alpha_2)/2}{(1 + \alpha_2)/2} & \frac{(-1 - \alpha_3)/2}{(-1 + \alpha_3)/2} \end{array} \right\},$$

and the equation (2.1b) to

$$(2.6b) \quad \left[D_t^2 + \left(\frac{1 - \alpha_1^2}{4t^2} + \frac{1 - \alpha_2^2}{4(t-1)^2} - \frac{1 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2}{4t(t-1)} \right) \right] f = 0.$$

This alternative normal form, self-adjoint in that it has a null coefficient function $P_1(t)$, is used, e.g., in conformal mapping. In the third special case of the PE, with $\mu_1\mu'_1 = \mu_2\mu'_2$, the P-symbol (2.1a) reduces to

$$(2.7a) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \frac{[(1 - \alpha_1 - \alpha_3)^2 - \alpha_2^2]/4(1 - \alpha_3)}{[(1 + \alpha_1 - \alpha_3)^2 - \alpha_2^2]/4(1 - \alpha_3)} & \frac{[(1 - \alpha_2 - \alpha_3)^2 - \alpha_1^2]/4(1 - \alpha_3)}{[(1 + \alpha_2 - \alpha_3)^2 - \alpha_1^2]/4(1 - \alpha_3)} & \begin{array}{c} 0 \\ \alpha_3 \end{array} \end{array} \right\},$$

and the equation (2.1b) to

$$(2.7b) \quad \left[D_t^2 + \left(\frac{1 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2}{2(1 - \alpha_3)t} + \frac{1 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2}{2(1 - \alpha_3)(t-1)} \right) D_t + \frac{\lambda}{16(1 - \alpha_3)^2 t^2 (1 - t)^2} \right] f = 0,$$

with $\lambda = [(1 - \alpha_3)^2 - (\alpha_1 + \alpha_2)^2][(1 - \alpha_3)^2 - (\alpha_1 - \alpha_2)^2]$. This third normal form is distinguished by its coefficient function $P_2(t)$ having only double poles.

By manipulation of P-symbols, the solutions of the general PE (2.1b) can be expressed in terms of those of the GHE (2.2b), such as ${}_2F_1$. Let f be a solution of (2.1b) and let $f_i = \Delta_i f$, $i = 1, 2, 3$, where

$$(2.8) \quad \Delta_i(t) := \left[-\frac{(t_i - t)(t_j - t_k)}{(t - t_j)(t_k - t_i)} \right]^{\mu_j} \left[-\frac{(t - t_i)(t_j - t_k)}{(t_k - t)(t_i - t_j)} \right]^{\mu_k} \\ \propto (t - t_i)^{\mu_j + \mu_k} (t - t_j)^{-\mu_j} (t - t_k)^{-\mu_k},$$

in which (i, j, k) is the cyclic permutation of $(1, 2, 3)$ determined by i . (Since the right side of (2.8) is expressed in terms of cross-ratios, f_i will be well-defined even if one of t_1, t_2, t_3 is taken to ∞ .) For each i , f_i will be a solution of the PE that has P-symbol

$$(2.9) \quad \left\{ \begin{array}{ccc} t_j & t_k & t_i \\ 0 & 0 & \mu_i + \mu_j + \mu_k \\ \mu'_j - \mu_j & \mu'_k - \mu_k & \mu'_i + \mu_j + \mu_k \end{array} \right\}.$$

To obtain a solution of a GHE from f_i , one must apply a Möbius transformation: $t \mapsto S_{ijk}(t) := (t - t_j)(t_k - t_i)/(t - t_i)(t_k - t_j)$, which takes t_j, t_k, t_i to $0, 1, \infty$. If

one defines F_{ijk} by $F_{ijk}(t) = f_i(S_{ijk}^{-1}(t))$, then $F_{ijk}(t)$ will be the solution of the PE with P-symbol

$$(2.10) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & \mu_i + \mu_j + \mu_k \\ \mu'_j - \mu_j & \mu'_k - \mu_k & \mu'_i + \mu_j + \mu_k \end{array} \right\},$$

i.e., of a certain GHE (cf. (2.2a)). Reversing this, one sees that if F_{ijk} is any solution of this GHE, such as the appropriate ${}_2F_1$, the function $f_i(t) = F_{ijk}(S_{ijk}(t))$ will be a solution of the PE with P-symbol (2.9), and furthermore that

$$(2.11) \quad f(t) = \Delta_i^{-1}(t) F_{ijk}(S_{ijk}(t))$$

will be a solution of the original PE (2.1b). By permuting i, j, k , one can generate $3! = 6$ local solutions of (2.1b) that are expressed in terms of ${}_2F_1$.

In a similar way, one can express solutions of any PE in terms of solutions of either of the remaining two normal forms of the PE, Eqs. (2.6b) and (2.7b). Each normal form is associated to a distinct triple Δ_i , $i = 1, 2, 3$, analogous to (2.8).

2.2. From PE to gDH. The following theorem facilitates the hypergeometric integration of gDH and DH systems. It also introduces an alternative parametrization of gDH systems, based not on $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ but on a birationally equivalent parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}; c)$.

Theorem 2.1. *Let $f = f(t)$ be a nonzero analytic local solution of a Papperitz equation of the form (2.1b) with exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$ satisfying Fuchs's relation $\sum_{i=1}^3 (\mu_i + \mu'_i) = 1$, and with (distinct) singular points $t_1, t_2, t_3 \in \mathbb{P}^1$. Define functions τ and x_1, x_2, x_3 of the local parameter t as follows.*

- (1) *For some $\bar{n} \in \mathbb{C} \setminus \{0\}$ and some 'offset vector' $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{C}^3$ satisfying $\kappa_1 + \kappa_2 + \kappa_3 = 1$, choose $\tau = \tau(t)$ to satisfy*

$$\frac{d\tau}{dt} = K^{-2}(t) f^{-1/\bar{n}}(t),$$

where

$$(2.12) \quad K^2(t) := \left[\frac{(t-t_1)^2(t_2-t_3)}{(t_1-t_2)(t_3-t_1)} \right]^{\kappa_1} \left[\frac{(t-t_2)^2(t_3-t_1)}{(t_2-t_3)(t_1-t_2)} \right]^{\kappa_2} \left[\frac{(t-t_3)^2(t_1-t_2)}{(t_3-t_1)(t_2-t_3)} \right]^{\kappa_3}.$$

- (2) *For some $c \in \mathbb{C} \setminus \{0\}$, define each x_i as a logarithmic derivative:*

$$\begin{aligned} x_i(t) &= c^{-1} \bar{n}^{-1} \frac{d}{d\tau} \log f_i \\ &= c^{-1} \frac{d}{d\tau} \log \left[K^{-2} \Delta_i^{1/\bar{n}} (d\tau/dt)^{-1} \right], \end{aligned}$$

where $f_i = \Delta_i f$ and

$$(2.13) \quad \Delta_i(t) := \left[-\frac{(t_i-t)(t_j-t_k)}{(t-t_j)(t_k-t_i)} \right]^{\mu_j} \left[-\frac{(t-t_i)(t_j-t_k)}{(t_k-t)(t_i-t_j)} \right]^{\mu_k},$$

in which (i, j, k) is the cyclic permutation of $(1, 2, 3)$ determined by i . Then $x = (x_1, x_2, x_3)$, viewed as a function of τ , will satisfy the generalized Darboux–Halphen system (1.4), $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, with parameters $(a_1, a_2, a_3; b_1, b_2, b_3)$ computed from $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3; \bar{n})$ and c , i.e., from $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$ and c ,

by

$$(2.14) \quad \begin{cases} a_i = \frac{\nu_i}{\nu_1 + \nu_2 + \nu_3 + \bar{n}} c, \\ b_i = \frac{-\bar{n}\nu'_i - (\bar{n} - 1)(\nu_j + \nu_k) - \bar{n}(\bar{n} - 1)}{\nu_1 + \nu_2 + \nu_3 + \bar{n}} c, \end{cases}$$

where

$$(2.15) \quad \nu_i := \mu_i + (2\kappa_i - 1)\bar{n}, \quad \nu'_i := \mu'_i + 2(\kappa_i - 1)\bar{n}$$

are offset exponents, satisfying $\sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n}$. It is assumed in this that the denominator $\rho^{-1} := \nu_1 + \nu_2 + \nu_3 + \bar{n}$ in (2.14) is nonzero. Equivalently, $(a_1, a_2, a_3; b_1, b_2, b_3)$ are determined implicitly by the inverse formulas

$$(2.16) \quad \begin{cases} \bar{n} = \frac{2c - b_1 - b_2 - b_3}{c}, \\ \nu_i = \bar{n} \left(\frac{a_i}{c - a_1 - a_2 - a_3} \right), \\ \alpha_i := \nu'_i - \nu_i = \frac{-c - a_i + b_j + b_k}{c - a_1 - a_2 - a_3}, \end{cases}$$

which express $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$ in terms of $(a_1, a_2, a_3; b_1, b_2, b_3)$ and c .

Remark. The formulas (2.12), (2.13) for K^2 and Δ_i (the latter based on cross-ratios) are perhaps overly elaborate: they were crafted to cover the case when one of t_1, t_2, t_3 is taken to ∞ . If each t_i is finite then

$$(2.17) \quad \begin{aligned} K(t) &\propto (t - t_1)^{\kappa_1} (t - t_2)^{\kappa_2} (t - t_3)^{\kappa_3}, \\ \Delta_i(t) &\propto (t - t_i)^{\mu_j + \mu_k} (t - t_j)^{-\mu_j} (t - t_k)^{-\mu_k}, \end{aligned}$$

the validity of the theorem being unaffected by the choice of proportionality constants. For instance, if $(t_1, t_2, t_3) = (0, 1, \infty)$ then

$$(2.18) \quad (\Delta_1, \Delta_2, \Delta_3) = ((-t)^{\mu_2 + \mu_3} (t - 1)^{-\mu_2}, (-t)^{-\mu_1} (t - 1)^{\mu_1 + \mu_3}, (-t)^{-\mu_1} (t - 1)^{-\mu_2}),$$

so that in this case, if $\mu_1 + \mu_2 + \mu_3 = \nu_1 + \nu_2 + \nu_3 + \bar{n} =: \rho^{-1}$ is nonzero, one can simply write

$$(2.19) \quad (\Delta_1^\rho, \Delta_2^\rho, \Delta_3^\rho) = \Delta_1^\rho \times (1, (1 - t)/t, -1/t),$$

$$(2.20) \quad (f_1^\rho, f_2^\rho, f_3^\rho) = f_1^\rho \times (1, (1 - t)/t, -1/t).$$

Also in this case, if $(\kappa_1, \kappa_2, \kappa_3) = (0, 0, 1)$ then K^2 will simply equal unity.

Remark. The expressions for Δ_i , $i = 1, 2, 3$, originate in the first normal form of the PE (the GHE); see (2.8). As $f_i = \Delta_i f$, each of f_1, f_2, f_3 satisfies a GHE. Thus the gDH system (1.4) for the vector of logarithmic derivatives $(x_1, x_2, x_3) =: x$ is associated specifically to the GHE. Alternative gDH systems (i.e., non-gDH HQDS's) associated to the other two normal forms of the PE can readily be derived; but up to linear equivalence, i.e. up to the action of $GL(3, \mathbb{C})$ on $x \in \mathbb{C}^3$, they will be no different.

Proof. For convenience, define the logarithmic derivatives

$$(2.21) \quad \begin{aligned} \bar{\kappa}(t) &= K_t/K = \kappa_1(t - t_1)^{-1} + \kappa_2(t - t_2)^{-1} + \kappa_3(t - t_3)^{-1}, \\ \delta_i(t) &= (\Delta_i)_t/\Delta_i = (\mu_j + \mu_k)(t - t_i)^{-1} - \mu_j(t - t_j)^{-1} - \mu_k(t - t_k)^{-1}. \end{aligned}$$

Also introduce an *ad hoc* notation for certain homogeneous quadratic polynomials,

$$(2.22) \quad [A_2, A_1, A_0] := \sum_{m=0}^2 A_m \left(K^2 f^{1/\bar{n}} \right)^{2-m} \left(\frac{f_\tau}{f} \right)^m.$$

By definition,

$$(2.23) \quad x_i = c^{-1} \bar{n}^{-1} \left(\frac{f_\tau}{f} + \delta_i K^2 f^{1/\bar{n}} \right),$$

so that

$$(2.24a) \quad (x_i - x_j)(x_k - x_i) = c^{-2} \bar{n}^{-2} [0, 0, (\delta_i - \delta_j)(\delta_k - \delta_i)],$$

$$(2.24b) \quad x_p x_q = c^{-2} \bar{n}^{-2} [1, \delta_p + \delta_q, \delta_p \delta_q].$$

The PE (2.1b) is of the form $[D_t^2 + P_1(t)D_t + P_2(t)]f = 0$, and changing the variable of differentiation from t to τ converts it to

$$(2.25) \quad \frac{f_{\tau\tau}}{f} + [-\bar{n}^{-1}, P_1 - 2\bar{\kappa}, P_2] = 0.$$

A similar manipulation of the formula for $\dot{x}_i = dx_i/d\tau$ yields

$$(2.26) \quad \dot{x}_i = c^{-1} \bar{n}^{-1} \left\{ \frac{f_{\tau\tau}}{f} + [-1, \bar{n}^{-1} \delta_i, (\delta_i)_t + 2\bar{\kappa} \delta_i] \right\},$$

which combined with (2.25) gives

$$(2.27) \quad \dot{x}_i = c^{-1} \bar{n}^{-1} [-1 + \bar{n}^{-1}, -P_1 + \bar{n}^{-1} \delta_i + 2\bar{\kappa}, -P_2 + (\delta_i)_t + 2\bar{\kappa} \delta_i].$$

The representations (2.24) and (2.27) reduce the question of the linear dependence of the local functions $(x_i - x_j)(x_k - x_i)$, $\{x_p x_q\}_{p,q=1}^3$, and \dot{x}_i to linear algebra over $\mathbb{C}(t)$, as each function is effectively a 3-vector $[A_2(t), A_1(t), A_0(t)]$ of rational functions of t . Each of the three equations in the gDH system (1.4) can thus be viewed as a relation of dependence among 3-vectors of rational functions of t , with $a_1, a_2, a_3; b_1, b_2, b_3; c$ appearing as coefficients. If $a_1, a_2, a_3; b_1, b_2, b_3$ are computed from the formulas in the theorem, the validity of each relation can be verified by hand, though the use of a computer algebra system is recommended. \square

Theorem 2.1 reveals that any ‘proper’ gDH system (to be defined shortly) can be integrated parametrically in terms of solutions of the GHE; and generically, in terms of the canonical hypergeometric function ${}_2F_1$. The integration proceeds as follows. Given $(a_1, a_2, a_3; b_1, b_2, b_3; c)$, one first uses the formulas (2.14) to compute $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$, and thus $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3; \bar{n})$, the exponent offset vector κ (satisfying $\kappa_1 + \kappa_2 + \kappa_3 = 1$) being chosen arbitrarily. Let f be a nonzero local solution of a PE (2.1b) with exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$, the choice of singular points (t_1, t_2, t_3) of the PE also being arbitrary. The PE having a 2-dimensional space of solutions, there are two free parameters in the choice of f , as one can write $f = K_1 f^{(1)} + K_2 f^{(2)}$; and f can generically be expressed in terms of ${}_2F_1$. The gDH

variables τ (independent) and x_1, x_2, x_3 (dependent) will be parametrized by t as

$$(2.28) \quad \begin{cases} \tau(t) = \tau_0 + \int^t \left(\frac{d\tau}{dt} \right) dt \\ \quad = \tau_0 + C \int^t (t-t_1)^{-2\kappa_1} (t-t_2)^{-2\kappa_2} (t-t_3)^{-2\kappa_3} f^{-1/\bar{n}}(t) dt, \\ x_i(t) = c^{-1} \bar{n}^{-1} \left(\frac{d\tau}{dt} \right)^{-1} \left(\frac{(f_i)_t}{f_i} \right), \end{cases}$$

where $f_i := \Delta_i f$, $i = 1, 2, 3$, are as in the theorem (they depend on μ_1, μ_2, μ_3), and C depends on t_1, t_2, t_3 . By transposing the ordered pairs (ν_i, ν'_i) and hence the ordered pairs (μ_i, μ'_i) , one can produce $2^3 = 8$ formally distinct integrations.

In (2.28) the formula for $\tau = \tau(t)$ contains an additional parameter, $\tau_0 \in \mathbb{C}$. Thus there are three free parameters in the local solution $t \mapsto (\tau; x_1, x_2, x_3)$. Hence any integration of this type will generate a 3-parameter family of solutions of the gDH system. PE-based integration is therefore *complete*: it constructs the general solution. (Special solutions with less than the full complement of three free parameters will be discussed below.) The freedom in the choice of τ_0 and the choice of scale of the PE solution f can be viewed as the cause of the covariance of the gDH system under affine transformations, i.e., under $\tau \mapsto A\tau + B$.

There is an obvious caveat. Theorem 2.1 assumes that $\bar{n} \neq \infty$ and $\bar{n} \neq 0$, i.e., that $c \neq 0$ and $2c - b_1 - b_2 - b_3 \neq 0$; also, that the calculation of the exponent parameters ν_i, ν'_i does not involve a division by zero, i.e., $c - a_1 - a_2 - a_3 \neq 0$.

Definition 2.2. A gDH system (1.4) is *proper* if its parameter vector $(a_1, a_2, a_3; b_1, b_2, b_3; c) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}$ satisfies (i) $c \neq 0$, (ii) $c - a_1 - a_2 - a_3 \neq 0$, and (iii) $2c - b_1 - b_2 - b_3 \neq 0$. Thus a proper DH system (see the Introduction) is a proper gDH system that satisfies $b_1 = b_2 = b_3 = c/2$.

Equivalently, a gDH system is proper if its alternative parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}; c) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C} \times \mathbb{C}$, constrained to satisfy $\sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n}$ and birationally related to $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ by (2.14), (2.16), satisfies (i) $c \neq 0$, (ii) the condition that

$$\rho^{-1} := \nu_1 + \nu_2 + \nu_3 + \bar{n} = (1 - \alpha_1 - \alpha_2 - \alpha_3)/2$$

be nonzero, where $\alpha_i := \nu'_i - \nu_i$, $i = 1, 2, 3$, are called the angular parameters of the gDH system, and (iii) $\bar{n} \neq 0$.

The just-described integration procedure can be applied to any *proper* gDH system. The system parameter ρ^{-1} (or its reciprocal ρ) will appear again, and it should be noted that

$$(2.29) \quad \rho = \frac{c - a_1 - a_2 - a_3}{2c - b_1 - b_2 - b_3}$$

when expressed in terms of $(a_1, a_2, a_3; b_1, b_2, b_3; c)$.

There is another caveat: it will shortly become clear that although PE-based integration is complete, it is not adapted to constructing special solutions in which two or more of x_1, x_2, x_3 coincide (necessarily, at all τ). But such solutions can often be found by examination. The solution (1.6), with $x_1 = x_2 = x_3$, has been noted. It is also easy to check that any gDH system (1.4) with $a_1 = 0$ and $2b_1 = b_2 + b_3 = c \neq 0$, such as a proper DH system $\text{DH}(0, \alpha_2, \alpha_3 | c)$, will have solution

$$(2.30) \quad x(\tau) = (c - b_1 - b_2 - b_3)^{-1} (\tau - \tau_*)^{-1} (1, 1, 1) + A(\tau - \tau_*)^{-2} (1, 0, 0)$$

for any $A \in \mathbb{C}$. This is a 2-parameter family of solutions with $x_2 = x_3$, the $A = 0$ case of which is (1.6).¹

Proposition 2.3. *In any gDH system (1.4), the elements $e = x_1e_1 + x_2e_2 + x_3e_3$ of the corresponding non-associative algebra \mathfrak{A} that satisfy $e * e \propto e$ include the following seven: $e_0; e_1, e_2, e_3; e'_1, e'_2, e'_3$, in which*

$$e_0 := e_1 + e_2 + e_3,$$

$$e'_i := (c - a_i - b_i)e_i + (-c - a_i + b_j + b_k)(e_j + e_k), \quad i = 1, 2, 3,$$

where j, k are the elements of $\{1, 2, 3\}$ other than i . If any such e is not nilpotent (i.e., $e * e \neq 0$), then it yields an idempotent $p \propto e$ of \mathfrak{A} , and a 1-parameter family of ray solutions of (1.4) along p , i.e., $x(\tau) = -(\tau - \tau_*)^{-1}p$. If e is nilpotent there is a ray of constant solutions instead, i.e., $x(\tau) \equiv Ke$, $K \in \mathbb{C}$.

Proof. Substitute $e = x_1e_1 + x_2e_2 + x_3e_3$ into $e * e = \lambda e$, the product $*$ coming from (1.1) and (1.4). If $\lambda \neq 0$ then e is idempotent; otherwise nilpotent. \square

Remark. The preceding illustrates the fact that any non-associative algebra over \mathbb{R} must have at least one idempotent or nilpotent [39]. Additionally, as any gDH algebra \mathfrak{A} has dimension $d = 3$ over \mathbb{C} , the number of its pairwise linearly independent idempotents and nilpotents, if finite, is $\leq 2^d - 1 = 2^3 - 1 = 7$. (See [66, Thm. 3.3].) For generic $(a_1, a_2, a_3; b_1, b_2, b_3; c)$, by examination there will be no nilpotents and exactly seven idempotents, namely $p_0; p_1, p_2, p_3; p'_1, p'_2, p'_3$, defined as above as scaled versions of $e_0; e_1, e_2, e_3; e'_1, e'_2, e'_3$. In nongeneric cases the latter may degenerate or become nilpotent.

Just as $e = e_0 := e_1 + e_2 + e_3$ yields the ray solution (1.6), for which $x_1 = x_2 = x_3$ at all τ , so do $e = e_i, e'_i$ yield respectively the ray solutions

$$(2.31a) \quad x(\tau) = -(\tau - \tau_*)^{-1}p_i = -a_i^{-1}(\tau - \tau_*)^{-1}e_i,$$

$$(2.31b) \quad x(\tau) = -(\tau - \tau_*)^{-1}p'_i = [c(c - a_i - b_j - b_k) + a_i(b_i + b_j + b_k)]^{-1}(\tau - \tau_*)^{-1}e'_i,$$

for which $x_j = x_k$ at all τ . (If the factor multiplying $e = e_i, e'_i$ diverges then e is nilpotent, and $x \equiv Ke$, $K \in \mathbb{C}$, is a ray of nilpotents.) The solutions (2.31) can be embedded in families like (2.30), which may or may not have three free parameters.

In the integration of differential equations, solutions without a full complement of free parameters are traditionally called ‘singular’ solutions [19]. They lie off the general-solution manifold, and may not be generated by a generic integration procedure. The following more restrictive definition will be useful here.

Definition 2.4. Any meromorphic local solution $x = x(\tau)$ of the gDH system (1.4) is *coincident* if two or more components are coincident (necessarily, at all τ).

The following two lemmas will be used below. The first (Lemma 2.5) is phrased so as to refer to PE-based integration, but its formulas will find broader application. The most important are (2.32) and (2.34), which construct $x = x(\tau)$ from $t = t(\tau)$, and reconstruct $t = t(\tau)$ from any noncoincident $x = x(\tau)$. Outside PE-based integration, the latter will be used as the *definition* of $t = t(\tau)$; and the two will be called the $t(\cdot) \mapsto x(\cdot)$ map and the $x(\cdot) \mapsto t(\cdot)$ map.

¹It generalizes a well-known 2-parameter family of solutions of Chazy-III [38]. There are only two free parameters because for the ray solution (1.6) of the gDH system (parameters constrained as stated), the set \mathcal{R} of Kovalevskaya exponents [27, 62] is degenerate: $\mathcal{R} = \{-1, -1, -1\}$.

Lemma 2.5. *As in Theorem 2.1, let $\tau = \tau(t)$ and $x = x(t)$ be computed from a nonzero local Papperitz solution $f = f(t)$, the latter via $x_i = c^{-1}\bar{n}^{-1}\dot{f}_i/f_i$, where the dot indicates $d/d\tau$. Then for $i = 1, 2, 3$, with j, k the elements other than i ,*

$$(2.32) \quad \begin{aligned} x_i &= c^{-1} \frac{d}{d\tau} \log \left[\frac{\dot{t}}{(t-t_i)^{-(\nu_j+\nu_k)/\bar{n}} (t-t_j)^{(\nu_j+\bar{n})/\bar{n}} (t-t_k)^{(\nu_k+\bar{n})/\bar{n}}} \right] \\ &= c^{-1} \left[\frac{\ddot{t}}{\dot{t}} - \bar{n}^{-1} \sum_{l=1}^3 (\nu_l + \bar{n}) \frac{\dot{t}}{t-t_l} \right] + c^{-1} \bar{n}^{-1} (\nu_1 + \nu_2 + \nu_3 + \bar{n}) \frac{\dot{t}}{t-t_i}, \end{aligned}$$

where $\nu_l := \mu_l + (2\kappa_l - 1)\bar{n}$ as usual, so $\nu_1 + \nu_2 + \nu_3 + \bar{n} = \mu_1 + \mu_2 + \mu_3$. Also,

$$(2.33) \quad x_i - x_j = c^{-1} \bar{n}^{-1} (\nu_1 + \nu_2 + \nu_3 + \bar{n}) \left[\frac{\dot{t}}{t-t_i} - \frac{\dot{t}}{t-t_j} \right]$$

for any $i, j \in \{1, 2, 3\}$. Furthermore, if x_1, x_2, x_3 do not coincide, then

$$(2.34) \quad t = - \left[\frac{t_1 t_2 (x_1 - x_2) + t_2 t_3 (x_2 - x_3) + t_3 t_1 (x_3 - x_1)}{t_1 (x_2 - x_3) + t_2 (x_3 - x_1) + t_3 (x_1 - x_2)} \right].$$

If $\rho^{-1} := \nu_1 + \nu_2 + \nu_3 + \bar{n}$ is nonzero then $f_1^\rho + f_2^\rho + f_3^\rho = 0$, and if x_1, x_2, x_3 do not coincide then

$$(2.35) \quad [x_2 - x_3 : x_3 - x_1 : x_1 - x_2] = [f_1^\rho : f_2^\rho : f_3^\rho],$$

interpreted as an equality between \mathbb{P}^2 -valued functions.

Proof. These four formulas are all consequences of the definitions of K^2 and Δ_i given in Theorem 2.1. By rewriting logarithmic derivatives,

$$(2.36) \quad \begin{aligned} x_i &= c^{-1} \bar{n}^{-1} (\dot{f}/f + \dot{\Delta}_i/\Delta_i) \\ &= c^{-1} \bar{n}^{-1} (\bar{n} \ddot{t}/\dot{t} - \bar{n} (K^2)^\cdot / K^2 + \dot{\Delta}_i/\Delta_i) \\ &= c^{-1} \ddot{t}/\dot{t} + c^{-1} [-(K^2)_t/K^2 + \bar{n}^{-1} (\Delta_i)_t/\Delta_i] \dot{t}, \end{aligned}$$

which leads to (2.32). Equation (2.33) is a corollary. It follows from (2.33) that

$$(2.37) \quad \frac{x_i - x_j}{x_j - x_k} = - \frac{(t_k - t)(t_i - t_j)}{(t - t_i)(t_j - t_k)},$$

and the formula (2.34) is obtained by solving (2.37) for t . One can check by hand that $\Delta_1^\rho + \Delta_2^\rho + \Delta_3^\rho = 0$, which implies $f_1^\rho + f_2^\rho + f_3^\rho = 0$; and that

$$(2.38) \quad [f_1^\rho : f_2^\rho : f_3^\rho] = [\Delta_1^\rho : \Delta_2^\rho : \Delta_3^\rho] \\ = [(t_2 - t_3)(t - t_1) : (t_3 - t_1)(t - t_2) : (t_1 - t_2)(t - t_3)].$$

Combining (2.33) with (2.38) yields (2.35). \square

Lemma 2.6. *For any noncoincident solution $x = x(\tau)$ of a proper gDH system of the form (1.4), if $t = t(\tau)$ is defined by the $x(\cdot) \mapsto t(\cdot)$ map (2.34), then*

$$(2.39) \quad \dot{t} = c \bar{n} \rho \left\{ \frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \cdot (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{[t_1(x_2 - x_3) + t_2(x_3 - x_1) + t_3(x_1 - x_2)]^2} \right\},$$

$$(2.40) \quad \frac{\ddot{t}}{\dot{t}} = c \rho \frac{\sum t_i (x_j - x_k) [(\nu_i - \bar{n})x_i + (\nu_j + \bar{n})x_j + (\nu_k + \bar{n})x_k]}{t_1(x_2 - x_3) + t_2(x_3 - x_1) + t_3(x_1 - x_2)},$$

the summation being over cyclic permutations i, j, k of $1, 2, 3$, and ρ being defined as usual by $\rho^{-1} := \nu_1 + \nu_2 + \nu_3 + \bar{n}$, or equivalently by (2.29). Additionally, the reconstruction formulas (2.32) and (2.33) for x_i and $x_i - x_j$ hold.

Remark. In the special case of a DH system with $(t_1, t_2, t_3) = (0, 1, \infty)$, the formulas (2.39), (2.40), and the consequent $t(\cdot) \mapsto x(\cdot)$ map (2.32), were previously obtained as Eqs. (17)–(20) of Ref. [1].

Proof. Equations (2.39), (2.40) can be verified by successively differentiating the gDH system (1.4). The formulas (2.32), (2.33) then follow from (2.34), (2.39), (2.40) by elementary manipulations.

It is worth noting that if $x = x(\tau)$ were constructed by PE-based integration, Eq. (2.39) would come by eliminating t from the expressions for $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$ provided by (2.33), and Eq. (2.40) by substituting (2.39) into (2.32), and solving for \ddot{t}/\dot{t} . \square

A ‘generalized Schwarzian’ equation satisfied by any function $t = t(\tau)$ derived from a noncoincident gDH solution $x = x(\tau)$ will now be introduced. This ODE is invariant under any affine transformation $\tau \mapsto \tau' := A\tau + B$. It will play a role in the Painlevé analysis of § 4.

Definition 2.7. The (triangular) generalized Schwarzian equation, called the gSE and denoted by $\text{gS}_{t_1, t_2, t_3}(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$, is the following nonlinear third-order equation for $t = t(\tau)$:

$$(2.41) \quad \frac{\ddot{t}}{\dot{t}^3} + (\bar{n} - 2) \left(\frac{\ddot{t}}{\dot{t}^2} \right)^2 + \left[\sum_{i=1}^3 \frac{1 - (\nu_i + \bar{n}) - (\nu'_i + \bar{n})}{t - t_i} \right] \frac{\ddot{t}}{\dot{t}^2} + \frac{1}{\bar{n}} \sum_{i=1}^3 \left[\frac{(\nu_i + \bar{n})(\nu'_i + \bar{n})}{(t - t_i)^2} + \frac{(\nu_i - \bar{n})(\nu'_i - \bar{n}) - (\nu_j + \bar{n})(\nu'_j + \bar{n}) - (\nu_k + \bar{n})(\nu'_k + \bar{n})}{(t - t_j)(t - t_k)} \right] = 0,$$

the subscripts j, k in the second summand being the elements of $\{1, 2, 3\}$ other than i . Here the vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}) \in (\mathbb{C}^2)^3 \times (\mathbb{C} \setminus \{0\})$ of parameters is required to satisfy the Fuchsian condition $\sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n}$. Also, $t_1, t_2, t_3 \in \mathbb{P}_t^1$ are distinct singular values, so that if $t_3 = \infty$ the gSE reduces to

$$(2.42) \quad \frac{\ddot{t}}{\dot{t}^3} + (\bar{n} - 2) \left(\frac{\ddot{t}}{\dot{t}^2} \right)^2 + \left[\sum_{i=1}^2 \frac{1 - (\nu_i + \bar{n}) - (\nu'_i + \bar{n})}{t - t_i} \right] \frac{\ddot{t}}{\dot{t}^2} + \frac{1}{\bar{n}} \left\{ \sum_{i=1}^2 \left[\frac{(\nu_i + \bar{n})(\nu'_i + \bar{n})}{(t - t_i)^2} \right] + \frac{(\nu_3 - \bar{n})(\nu'_3 - \bar{n}) - (\nu_1 + \bar{n})(\nu'_1 + \bar{n}) - (\nu_2 + \bar{n})(\nu'_2 + \bar{n})}{(t - t_1)(t - t_2)} \right\} = 0,$$

The gSE is invariant under each transposition $\nu_i \leftrightarrow \nu'_i$.

Theorem 2.8. Let $x = x(\tau)$ be a noncoincident local meromorphic solution of a proper gDH system $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, and let $t = t(\tau)$ be computed from $x = x(\tau)$ by Eq. (2.34), for some choice of distinct $t_1, t_2, t_3 \in \mathbb{P}_t^1$. (E.g., if (t_1, t_2, t_3) is $(0, 1, \infty)$ then $t = -(x_2 - x_3)/(x_1 - x_2)$.) Then t will satisfy a gSE $\text{gS}_{t_1, t_2, t_3}(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$, where the parameters, satisfying $\sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n}$, are computed from $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ by the formulas (2.16).

Proof. When $(t_1, t_2, t_3) = (0, 1, \infty)$, the gSE (2.41) can readily be derived by manipulating the gDH HQDS (1.4). That Eq. (2.41) in the reduced form (2.42) is satisfied by the Brioschi variable $t = -(x_2 - x_3)/(x_1 - x_2)$ in any proper gDH system with $(t_1, t_2, t_3) = (0, 1, \infty)$, satisfying $b_1 = b_2 = b_3$, was proved by Bureau [9]; and the restriction $b_1 = b_2 = b_3$ can be relaxed.

The case when t_1, t_2, t_3 are in general position, and t is obtained by the more general $x(\cdot) \mapsto t(\cdot)$ map, Eq. (2.34), also follows from (1.4). (Using a computer algebra system is recommended.) Verification is facilitated by the formulas for $\dot{t}, \ddot{t}/\dot{t}$ given in Lemma 2.6. Alternatively, one can simply show that Eq. (2.41) is reduced to (2.42) by the unique Möbius transformation of \mathbb{P}_t^1 that maps (t_1, t_2, t_3) to (t_1, t_2, ∞) , and in particular to $(0, 1, \infty)$. \square

Remark. A closely related claim dealing with PE-based integration, namely that if $\tau = \tau(t)$ is computed from a nonzero local PE solution $f = f(t)$ by the procedure of Theorem 2.1, then the locally defined inverse function $t = t(\tau)$ must satisfy the gSE (2.41), can also be proved. This is an exercise in calculus; cf. the derivation of Eq. (2.32). Starting with the PE (2.1b) satisfied by f , one uses $\dot{t} = K^2(t)f^{1/\bar{n}}(t)$ to convert each $d/d\tau$ to a d/dt , and substitutes the definition (2.12) for $K^2 = K^2(t)$. After much manipulation one ends up with a gSE satisfied by $t = t(\tau)$, including the Fuchsian condition, which in this derivation comes from Fuchs's relation $\sum_{i=1}^3(\mu_i + \mu'_i) = 1$ on the exponents of the PE. Details are left to the reader. A significant feature of this alternative derivation is that unlike the initial (linear) PE for $f = f(t)$, the (nonlinear) gSE (2.41) involves only the *offset* exponents $\nu_1, \nu_2, \nu_3; \nu'_1, \nu'_2, \nu'_3$. The offset vector κ does not appear explicitly. Of course if one integrates the gSE by substituting $\dot{t} = K^2(t)f^{1/\bar{n}}(t)$, with $K^2(t)$ defined as in Theorem 2.1, then $\kappa_1, \kappa_2, \kappa_3$ will reappear at least briefly.

Theorem 2.8 provides the forward half of a (slightly restricted) $\text{gDH} \leftrightarrow \text{gSE}$ correspondence, since it computes a local gSE solution $t = t(\tau)$ from any local *noncoincident* solution of any *proper* gDH system. The reverse direction $t(\cdot) \mapsto x(\cdot)$ is provided by Eq. (2.32), on account of the final sentence of Lemma 2.6. The proper $\text{gDH} \leftrightarrow \text{gSE}$ correspondence is summarized as follows.

Theorem 2.9. *For any proper gDH system $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, at any point in the τ -plane there is a bijection $x \leftrightarrow t$ between (i) its meromorphic local solutions $x = x(\tau)$ that are noncoincident and (ii) the meromorphic local solutions $t = t(\tau)$ of an associated gSE of the form (2.41). In the latter, the parameters $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$ are computed from $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ by the formulas (2.16). The maps $t(\cdot) \mapsto x(\cdot)$, $x(\cdot) \mapsto t(\cdot)$ of the bijection are given by Eqs. (2.32), (2.34). In all of this, the choice of (distinct) $t_1, t_2, t_3 \in \mathbb{P}^1$ is arbitrary. (E.g., if (t_1, t_2, t_3) is $(0, 1, \infty)$ then $t = -(x_2 - x_3)/(x_1 - x_2)$.)*

If in disagreement with the hypotheses of the theorem, the solution $x = x(\tau)$ is coincident (i.e., has coincident components), then the \mathbb{P}^1 -valued function $t = t(\tau)$ computed from (2.34) will either be undefined (if $x_1 = x_2 = x_3$), or be a constant: $t \equiv t_1, t_2$, or t_3 . (The values $t = t_1, t_2, t_3$ correspond to the invariant planes $x_2 - x_3 = 0$, $x_3 - x_1 = 0$, $x_1 - x_2 = 0$.) But by convention, a function $t \equiv \text{const}$ cannot be a solution of a Schwarzian equation such as the gSE (2.41), on account of divisions by zero.

Theorem 2.9 focuses on meromorphic local solutions of the gSE, but of course for a generic choice of $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$, generic gSE solutions $t = t(\tau)$ will have movable branch points. In the Painlevé analysis of § 4, such points will be ruled out by restricting the parameter vector. By substituting the formal statement $t = t(\tau) \sim t_* + C(\tau - \tau_*)^p$ into the gSE, it is easy to verify the following. If $t_* \neq t_1, t_2, t_3$ then $p \in \{\pm 1, \pm(n+1)\}$, with the statement holding as $\tau \rightarrow \tau_*$ if $\text{Re } p > 0$ and as $\tau \rightarrow \infty$ if $\text{Re } p < 0$. In this, $n := 1/(\bar{n} - 1)$ and $\bar{n} = (n+1)/n$. For the

gSE to have the Painlevé property (PP), one expects the restriction that n be an integer or ∞ . Also, if $t_* = t_i$ for one of $i = 1, 2, 3$, one finds that $p \in \{r_i, r'_i\}$, where $(r_i, r'_i) := -\bar{n}(1/\nu_i, 1/\nu'_i)$. As will be seen, for the gSE to have the PP, the exponents r_i, r'_i (if defined and finite) will also need to be integers.

2.3. Integrating DH systems. Any proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ of the form (1.7) is by definition a proper gDH system, with parameters

$$(2.43) \quad a_i = -\alpha_i c / (1 - \alpha_1 - \alpha_2 - \alpha_3), \quad i = 1, 2, 3,$$

and $b_1 = b_2 = b_3 = c/2$. Moreover, it follows from the birational correspondence (2.14), (2.16) that the alternative gDH parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}; c)$ equals $(-\alpha_1/2, \alpha_1/2; -\alpha_2/2, \alpha_2/2; -\alpha_3/2, \alpha_3/2; 1/2; c)$.

Thus an integration scheme for proper DH systems follows from the Papperitz-based scheme of § 2.2 for proper gDH systems, by setting $\bar{n} = 1/2$. This DH-specific value for \bar{n} is quite special: it permits the use of a *ratio* of a pair of PE solutions, and in fact of a pair of Gauss hypergeometric functions.

Lemma 2.10. *Let $\bar{n} = 1/2$ and let the offset vector κ be $\kappa_i = (1 - \mu_i - \mu'_i)/2$, $i = 1, 2, 3$. Then the definition of $\tau = \tau(t)$ given in Theorem 2.1 is equivalent to*

$$\tau(t) = f^{(1)}(t)/f^{(2)}(t),$$

where $f^{(1)}, f^{(2)}$ are any independent analytic local solutions of the Papperitz equation. Also, the proper gDH system produced by the theorem has $b_1 = b_2 = b_3 = c/2$, and thus reduces to a proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$.

Proof. The derivative of any ratio of solutions $f^{(1)}(t)/f^{(2)}(t)$ with respect to t equals $W(t)/[f^{(2)}(t)]^2$, where $W(t)$ is the Wronskian $f_t^{(1)}f^{(2)} - f_t^{(2)}f^{(1)}$. The PE (2.1b) is of the form $[D_t^2 + Q(t)D_t + Q(t)]f = 0$, so that

$$(2.44) \quad W(t) = \exp - \int^t Q(t) dt \propto (t - t_1)^{1-\mu_1-\mu'_1} (t - t_2)^{1-\mu_2-\mu'_2} (t - t_3)^{1-\mu_3-\mu'_3},$$

where the proportionality constant depends on the choice of $f^{(1)}, f^{(2)}$. If $\bar{n} = 1/2$, the condition in Theorem 2.1 that τ is defined to satisfy is that $d\tau/dt = K^{-2}/f^2$, where f is a nonzero solution of the PE and

$$(2.45) \quad K(t) \propto (t - t_1)^{\kappa_1} (t - t_2)^{\kappa_2} (t - t_3)^{\kappa_3},$$

the proportionality constant being a function of t_1, t_2, t_3 . By the preceding, when $\kappa_i = (1 - \mu_i - \mu'_i)/2$ this condition will be satisfied by the choice $\tau = f^{(1)}/f^{(2)}$, if f is a suitably scaled version of $f^{(2)}$. Also,

$$(2.46) \quad \begin{aligned} \nu_i &= \mu_i + (2\kappa_i - 1)\bar{n} = \mu_i + \kappa_i - 1/2 = (\mu_i - \mu'_i)/2 = -\alpha_i/2, \\ \nu'_i &= \mu'_i + (2\kappa_i - 1)\bar{n} = \mu'_i + \kappa_i - 1/2 = (\mu'_i - \mu_i)/2 = +\alpha_i/2. \end{aligned}$$

That $b_1 = b_2 = b_3 = c/2$ follows by substituting these expressions (and $\bar{n} = 1/2$) into the formula (2.14) for b_i . \square

In the $\bar{n} = 1/2$ case the integration procedure for proper gDH systems based on Theorem 2.1 specializes to the following procedure for proper DH systems, when enhanced by Lemma 2.10. (The formulas for μ_i, μ'_i are as in (2.46).)

Theorem 2.11. *Any proper Darboux–Halphen system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ of the form (1.7) can be integrated parametrically as follows. For any exponent offset vector $\kappa \in \mathbb{C}^3$ with $\kappa_1 + \kappa_2 + \kappa_3 = 1$, let*

$$\mu_i = (1 - \alpha_i)/2 - \kappa_i, \quad \mu'_i = (1 + \alpha_i)/2 - \kappa_i,$$

and let $f^{(1)}, f^{(2)}$ be any pair of independent analytic local solutions of a Papperitz equation (2.1b) with exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$, and (distinct) singular points $t_1, t_2, t_3 \in \mathbb{P}^1$. The independent and dependent variables, τ and x_i , $i = 1, 2, 3$, are then defined as functions of the local parameter t by

$$\tau(t) = f^{(1)}(t)/f^{(2)}(t),$$

$$x_i(t) = (2/c) \frac{d}{d\tau} \log f_i,$$

where $f_i = \Delta_i f^{(2)}$ and

$$\Delta_i(t) := \left[-\frac{(t_i - t)(t_j - t_k)}{(t - t_j)(t_k - t_i)} \right]^{\mu_j} \left[-\frac{(t - t_i)(t_j - t_k)}{(t_k - t)(t_i - t_j)} \right]^{\mu_k}.$$

In this formula, i, j, k is any cyclic permutation of $1, 2, 3$.

Remark. The factors Δ_i and the functions f_i are defined even if one of t_1, t_2, t_3 equals ∞ . As in Theorem 2.1, if $(t_1, t_2, t_3) = (0, 1, \infty)$ and $\mu_1 + \mu_2 + \mu_3 =: \rho^{-1}$, i.e., $\rho = 2/(1 - \alpha_1 - \alpha_2 - \alpha_3)$, the f_i will satisfy $(f_1^\rho, f_2^\rho, f_3^\rho) = f_1^\rho \times (1, (1-t)/t, -1/t)$.

This parametric integration of any proper DH system is complete, since by examination it includes three free parameters. There are three (rather than four) in the choice of $f^{(1)}, f^{(2)}$, and hence in $t \mapsto (\tau; x_1, x_2, x_3)$, as multiplying $f^{(1)}, f^{(2)}$ by a common factor affects neither $\tau = \tau(t)$ nor $x = x(t)$.

The integration can be ‘customized’ in the sense that the offset vector κ can be chosen to make the exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$ agree with those of any of the three normal-form P-symbols: (2.2a), (2.6a), and (2.7a). That is, the Papperitz equation can be chosen to agree with any of the normal forms (2.2b), (2.6b), and (2.7b) of the PE, if one also chooses $(t_1, t_2, t_3) = (0, 1, \infty)$. By examination, the first two of the respective offset vectors are $(1 - \alpha_1, 1 - \alpha_2, \alpha_1 + \alpha_2)/2$ and $(0, 0, 1)$. But when the solutions $f^{(1)}, f^{(2)}$ of the normal-form PE are expressed in terms of the Gauss hypergeometric function ${}_2F_1$, the same formulas for $\tau = \tau(t)$ and $x = x(t)$ in terms of ${}_2F_1(t)$ will always result. Details are left to the reader.

Like the parametric integration in §2.2 of a proper gDH system, this integration of a proper DH system constructs the general solution but is not adapted to constructing nongeneric, special solutions: coincident ones, in the sense of Definition 2.4. The ray solutions (1.6) and (2.31), each proportional to $(\tau - \tau_*)^{-1}$ and having coincident components, have been mentioned. However, since any proper DH system is a proper gDH system with $\nu_i = -\alpha_i/2$, $\nu'_i = \alpha_i/2$, and $\bar{n} = 1/2$, a specialization of Theorem 2.9 on the proper gDH \leftrightarrow gSE correspondence will hold. In the DH case the gSE specializes to what will be called the SE.

Definition 2.12. The (triangular) Schwarzian equation, called the SE and denoted by $S_{t_1, t_2, t_3}(\alpha_1, \alpha_2, \alpha_3)$, is the following nonlinear third-order equation for $t = t(\tau)$: (2.47)

$$\{\tau, t\} := -\frac{\ddot{t}}{t^3} + \frac{3}{2} \left(\frac{\ddot{t}}{t^2} \right)^2 = \frac{1}{2} \sum_{i=1}^3 \left[\frac{1 - \alpha_i^2}{(t - t_i)^2} + \frac{(1 - \alpha_i^2) - (1 - \alpha_j^2) - (1 - \alpha_k^2)}{(t - t_j)(t - t_k)} \right],$$

the subscripts j, k in the summand being the elements of $\{1, 2, 3\}$ other than i . Here $t_1, t_2, t_3 \in \mathbb{P}^1$ are distinct singular values, $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ is a vector of angular parameters, and the left side $\{\tau, t\} = -\{t, \tau\}/t^2$ is the standard Schwarzian derivative of τ with respect to t . The SE is invariant under each negation $\alpha_i \mapsto -\alpha_i$.

Also, in the case of a proper DH (rather than merely a proper gDH) system, the $t(\cdot) \mapsto x(\cdot)$ map (2.32) specializes to

$$(2.48) \quad \begin{aligned} x_i &= c^{-1} \frac{d}{d\tau} \log \left[\frac{\dot{t}}{(t-t_i)^{\alpha_j+\alpha_k} (t-t_j)^{1-\alpha_j} (t-t_k)^{1-\alpha_k}} \right] \\ &= c^{-1} \left[\frac{\ddot{t}}{\dot{t}} - \sum_{l=1}^3 (1-\alpha_l) \frac{\dot{t}}{t-t_l} \right] + c^{-1} (1-\alpha_1-\alpha_2-\alpha_3) \frac{\dot{t}}{t-t_i}. \end{aligned}$$

The following is the specialization of Theorem 2.9 to the DH case.

Theorem 2.13. *For any proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$, at any point in the τ -plane there is a bijection $x \leftrightarrow t$ between (i) its meromorphic local solutions $x = x(\tau)$ that are noncoincident and (ii) the meromorphic local solutions $t = t(\tau)$ of an associated SE of the form (2.47). The maps $t(\cdot) \mapsto x(\cdot)$, $x(\cdot) \mapsto t(\cdot)$ of the bijection are given by Eqs. (2.48), (2.34). In all of this, the choice of (distinct) $t_1, t_2, t_3 \in \mathbb{P}^1$ is arbitrary. (E.g., if (t_1, t_2, t_3) is $(0, 1, \infty)$ then $t = -(x_2 - x_3)/(x_1 - x_2)$.)*

The proper gDH \leftrightarrow gSE correspondence thus specializes to a proper DH \leftrightarrow SE correspondence.

The SE (2.47) is familiar from conformal mapping theory [54]. It is satisfied by any meromorphic function $t = t(\tau)$ that maps the interior of a hyperbolic triangle with angles $\pi(\alpha_1, \alpha_2, \alpha_3)$ in the complex τ -plane, where $\alpha_i \geq 0$ for all i with $\alpha_1 + \alpha_2 + \alpha_3 < 1$, to a disk in the t -plane that is bounded by the circle through t_1, t_2, t_3 . It is most familiar in the case when t_1, t_2, t_3 are taken to be the collinear points $0, 1, \infty$, and the disk becomes the upper half t -plane.

When $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$, where each N_i is a positive integer or ∞ , with $\alpha_1 + \alpha_2 + \alpha_3 < 1$, a standard result from conformal mapping theory applies. In this case the function t (and hence each x_i , by (2.48)) can be extended from the hyperbolic triangle in the τ -plane by the Schwarz reflection principle, i.e. by reflecting the triangle and its images repeatedly through their sides, thereby tessellating a larger domain: the interior of a disk or half-plane. By definition, the group of reflections acting on this domain is a triangle group $\Delta(N_1, N_2, N_3) < PSL(2, \mathbb{C})$, under which t (though not x_1, x_2, x_3) will be automorphic, i.e., invariant.

As will be explained in Part II, in the upper half-plane case the logarithmic derivatives x_1, x_2, x_3 can be viewed as quasi-modular forms, with an affine-linear transformation law under $\Delta(N_1, N_2, N_3) < PSL(2, \mathbb{R})$ that resembles Eq. (2.49) below. For several choices of (N_1, N_2, N_3) for which $\Delta(N_1, N_2, N_3)$ is an arithmetic group, it will be possible to express each x_i as an explicit q -series.

The Schwarzian derivative $\{\tau, t\}$ is invariant under Möbius transformations of τ , which implies that the SE (2.47) is invariant under a larger group than is the gSE (2.41): not merely under affine transformations of the independent variable, but under any invertible point transformation $\tau \mapsto \tau' = (A\tau + B)/(C\tau + D)$. By the $t(\cdot) \mapsto x(\cdot)$ map (2.48), this $\tau \mapsto \tau'$ induces a transformation of the solution $x = x(\tau)$ of the associated proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$. But the transformation can be investigated without reference to the SE (2.47), as follows.

Theorem 2.14. *Any (not necessarily proper) Darboux–Halphen system $\dot{x} = Q(x)$, denoted by $\text{DH}(a_1, a_2, a_3; c)$, has a 3-parameter group of Lie point symmetries: for any $\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$, the system is stabilized by a point transformation of $\mathbb{C} \times \mathbb{C}^3 \ni (\tau; x)$ that faithfully represents $\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, namely*

$$(2.49) \quad \begin{aligned} \tau &\mapsto \tau' = (A\tau + B)/(C\tau + D), \\ x_i &\mapsto x'_i = (C\tau + D)^2 x_i + (c/2) C(C\tau + D), \quad i = 1, 2, 3. \end{aligned}$$

The corresponding infinitesimal statement is that the normalizer $\mathcal{N}(v)$ of the dynamical vector field $v = \partial_\tau + \sum_{i=1}^3 Q_i(x_1, x_2, x_3) \partial_{x_i}$ on $\mathbb{C} \times \mathbb{C}^3$ contains the 3-dimensional Lie algebra of vector fields generated by v_0, v_1, v_2 , given by

$$\partial_\tau, \quad \tau \partial_\tau - \sum_{i=1}^3 x_i \partial_{x_i}, \quad (c/2) \tau^2 \partial_\tau - \sum_{i=1}^3 (c \tau x_i + 1) \partial_{x_i}$$

and having commutators

$$[v_0, v_1] = v_0, \quad [v_0, v_2] = c v_1, \quad [v_1, v_2] = v_2,$$

which for $c \neq 0$, e.g. for a proper DH system, is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Proof. Invariance of any DH system, i.e., any gDH system (1.4) satisfying $b_1 = b_2 = b_3 = c/2$, under the stated action $(x, \tau) \mapsto (x', \tau')$ of any $\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$, can be verified by direct computation. A closely related fact should be mentioned: any parametric solution $(\tau, x) = (\tau(t), x(t))$ of $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$, constructed by the integration scheme of Theorem 2.11, is mapped to another such solution, as one sees by considering the effects of

$$\begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \mapsto \pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix},$$

taking into account that $\tau(t) = f^{(1)}(t)/f^{(2)}(t)$.

Each of the vector fields v_0, v_1, v_2 is an infinitesimal symmetry on $\mathbb{C} \times \mathbb{C}^3 \ni (\tau; x)$, because each generates a 1-parameter subgroup of $\text{PSL}(2, \mathbb{C})$. Alternatively, that $v_i \in \mathcal{N}(v)$ for each i can be verified directly: the Lie bracket (i.e., commutator) of each v_i with v turns out to be a scalar multiple of v . \square

Any gDH system is invariant under v_0 and v_i , the first two of the three infinitesimal Lie point transformations above, because it is stabilized by translation of τ , and by $\tau \mapsto \tau/\lambda$ when accompanied by $x \mapsto \lambda x$. That is, it is covariant under affine transformations. The point of the theorem is that in the DH case, the system is also covariant under *projective* transformations of τ (infinitesimal or otherwise).

What is remarkable is that although a non-DH gDH system may have only a 2-parameter group of Lie point symmetries, if proper it will nonetheless be integrable by the Papperitz-based integration scheme of § 2.2.

For any proper DH system, the existence of a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ is reminiscent of the Chazy-III equation. The 3-dimensionality of the Chazy-III symmetry algebra facilitates integration, though $\mathfrak{sl}(2, \mathbb{C})$ is not solvable and the usual Lie integration technique cannot be used. (See [15], and also [37].) The integration of $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$ extends that of Chazy-III, as will be seen in Part II.

2.4. Alternative DH systems. It is useful to compare the standardized proper DH system (1.7), which appeared for the first time in [49], to other proper DH systems in the literature. In 1881, Halphen [31] used ${}_2F_1$ (or equivalently, theta constants) to integrate a 3-dimensional HQDS that is identical to (1.7), up to a trivial reparametrization. Halphen's system was a generalization of a system introduced in 1878, in Darboux's analysis of triply orthogonal surfaces. (Darboux's system had $\alpha_1 = \alpha_2 = \alpha_3 = 0$.) Halphen's system has been the basis of several recent papers, e.g., [32].

An alternative form of Halphen's system has appeared in other recent papers (e.g., [2, 55, 63]). This form is

$$(2.50) \quad \begin{cases} \dot{y}_1 = y_1^2 + (y_1 - y_2)(y_3 - y_1) - T^2, \\ \dot{y}_2 = y_2^2 + (y_2 - y_3)(y_1 - y_2) - T^2, \\ \dot{y}_3 = y_3^2 + (y_3 - y_1)(y_2 - y_3) - T^2, \\ T^2 := \sum_{(i,j,k)} \alpha_i^2 (y_i - y_j)(y_k - y_i), \end{cases}$$

the summation being over cyclic permutations of $(1, 2, 3)$. The system (2.50) arises naturally in the symmetry reduction of the self-dual Yang–Mills equations [2], and differs from (1.7) except in the Darboux case $\alpha_1 = \alpha_2 = \alpha_3 = 0$. But the two systems are equivalent under $GL(3, \mathbb{C})$. The following is an exercise in linear algebra.

Proposition 2.15. *For any $(k_1, k_2, k_3) \in \mathbb{C}^3$ with $k_1 + k_2 + k_3 \neq 1$, define $y = (y_1, y_2, y_3)$ in terms of $x = (x_1, x_2, x_3)$, and vice versa, by*

$$\begin{cases} y_1 = (1 - k_2 - k_3)x_1 + k_2x_2 + k_3x_3, \\ y_2 = k_1x_1 + (1 - k_3 - k_1)x_2 + k_3x_3, \\ y_3 = k_1x_1 + k_2x_2 + (1 - k_1 - k_2)x_3 \end{cases}$$

and

$$\begin{cases} x_1 = (1 - k_1 - k_2 - k_3)^{-1} [(1 - k_1)y_1 - k_2y_2 - k_3y_3], \\ x_2 = (1 - k_1 - k_2 - k_3)^{-1} [-k_1y_1 + (1 - k_2)y_2 - k_3y_3], \\ x_3 = (1 - k_1 - k_2 - k_3)^{-1} [-k_1y_1 - k_2y_2 + (1 - k_3)y_3]. \end{cases}$$

Then $x = x(\tau)$ will satisfy a proper Darboux–Halphen system $\text{DH}(\alpha_1, \alpha_2, \alpha_3)$ in the form (1.7), with the normalization $c = 2$, if and only if $y = y(\tau)$ satisfies

$$\begin{cases} \dot{y}_1 = y_1^2 + (y_1 - y_2)(y_3 - y_1) - T^2, \\ \dot{y}_2 = y_2^2 + (y_2 - y_3)(y_1 - y_2) - T^2, \\ \dot{y}_3 = y_3^2 + (y_3 - y_1)(y_2 - y_3) - T^2, \end{cases}$$

where, with $\rho := 2/(1 - \alpha_1 - \alpha_2 - \alpha_3)$ as usual,

$$T^2 := \sum_{(i,j,k)} \frac{-k_i(k_i + \rho\alpha_i)}{(1 - k_1 - k_2 - k_3)^2} (y_i - y_j)(y_k - y_i).$$

The alternative DH system (2.50) comes from this lemma by letting $(k_1, k_2, k_3) = -(\rho/2)(\alpha_1, \alpha_2, \alpha_3)$.

Our choice of (1.7) as the standardized proper DH system is to an extent arbitrary. In principle one could choose the system (2.50), or any system produced by

Prop. 2.15. (The corresponding non-associative algebras \mathfrak{A} are of course isomorphic.) But (1.7) has several advantages. It is uncomplicated and easy to write; it extends to the gDH system (1.4), which is itself a slight extension of a well-known system of Bureau [9, 10]; it is naturally associated to the (nice) GHE normal form of the PE, rather than to any other form; and moreover it appears in the theory of ODE's satisfied by modular forms (see [49] and Part II).

3. RATIONAL MORPHISMS OF GDH SYSTEMS

3.1. Goals and context. The chief result obtained in this section is Theorem 3.1, accompanied by Tables 3 and 4, and Figure 1. There is a rich collection of nonlinear but *rational* solution-preserving maps between gDH systems of the form (1.4), generally with different parameters. The maps follow from the PE-based gDH integration scheme of § 2.2. They come from liftings of PE's to other PE's, or equivalently, from liftings of generalized Schwarzian equations (gSE's) to other gSE's.

Each such rational map, from a gDH system $(\mathbb{C}^3, \dot{\tilde{x}} = \tilde{Q}(\tilde{x}))$ to another gDH system $(\mathbb{C}^3, \dot{x} = Q(x))$, is a rational function $x = \Phi(\tilde{x})$. As such, it has a singular locus in $\mathbb{C}^3 \ni \tilde{x}$, on which at least one of x_1, x_2, x_3 diverges. The map Φ takes each local solution $\tilde{x} = \tilde{x}(\tau)$ of $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ (beginning off the singular locus, and not crossing it) to a solution $x = x(\tau)$ of $\dot{x} = Q(x)$.

The simplest example is the quadratic transformation $x = \Phi(\tilde{x})$ given by

$$(3.1a) \quad x_1 = \frac{1}{2}(\tilde{x}_1 + \tilde{x}_3), \quad x_2 = \tilde{x}_2, \quad x_3 = \frac{\tilde{x}_2(\tilde{x}_1 + \tilde{x}_3) - 2\tilde{x}_1\tilde{x}_3}{2\tilde{x}_2 - (\tilde{x}_1 + \tilde{x}_3)},$$

the singular locus of which is the plane $2\tilde{x}_2 - (\tilde{x}_1 + \tilde{x}_3) = 0$. This map Φ is solution-preserving when the parameter vector of $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, namely $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$, satisfies $\tilde{a}_3 = \tilde{a}_1$, $\tilde{b}_3 = \tilde{b}_1$, and the parameter vector of $\dot{x} = Q(x)$, say $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ with $c = \tilde{c}$, is computed from it by

$$(3.1b) \quad \begin{aligned} a_1 &= 2\tilde{a}_1, & a_2 &= \tilde{a}_2, & a_3 &= 2\tilde{a}_1 + \tilde{a}_2 - \tilde{c}, \\ b_1 &= \tilde{c} - \tilde{b}_2, & b_2 &= \tilde{b}_2, & b_3 &= 2\tilde{b}_1 + \tilde{b}_2 - \tilde{c}. \end{aligned}$$

One can view (3.1a) as a degree-2 map from the projective plane \mathbb{P}_x^2 , on which $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are homogeneous coordinates, to the projective plane \mathbb{P}_x^2 , i.e., as

$$\begin{aligned} \Phi: [\tilde{x}_1 : \tilde{x}_2 : \tilde{x}_3] &\mapsto [x_1 : x_2 : x_3] := \\ &[(\tilde{x}_1 + \tilde{x}_3)(\tilde{x}_1 + \tilde{x}_3 - 2\tilde{x}_2) : 2\tilde{x}_2(\tilde{x}_1 + \tilde{x}_3 - 2\tilde{x}_2) : -2(\tilde{x}_2(\tilde{x}_1 + \tilde{x}_3) - 2\tilde{x}_1\tilde{x}_3)]. \end{aligned}$$

A notable feature of (3.1a) is that if no two of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ coincide, then the quadruple $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, x_3$ will be equianharmonic: it will have cross-ratio 2.

It is difficult to construct solution-preserving maps between HQDS's, though they can be treated abstractly: Röhl [58, § 5] defines a category of which the objects are HQDS's and the morphisms are solution-preserving maps that are germs of analytic functions. By definition, any solution-preserving Φ from $(\mathbb{C}^3, \dot{\tilde{x}} = \tilde{Q}(\tilde{x}))$ to $(\mathbb{C}^3, \dot{x} = Q(x))$ satisfies $Q = \Phi_*\tilde{Q}$, i.e., the 'determining equations'

$$(3.2) \quad Q(\Phi(\tilde{x})) = [D\Phi(\tilde{x})] \tilde{Q}(\tilde{x}),$$

a system of coupled nonlinear PDE's for which there exists no solution algorithm.

For \tilde{Q}, Q of gDH type with suitably constrained parameter values, we shall construct rational solutions Φ of (3.2), each of which can be viewed as a $\mathbb{P}_x^2 \rightarrow \mathbb{P}_x^2$ map, by exploiting the Papperitz-based parametric integration of gDH systems, as

developed in §2.2. Each Φ will come from a *hypergeometric transformation*: a lifting of a Papperitz equation (PE) on \mathbb{P}_t^1 to a PE on a covering \mathbb{P}_t^1 , along a rational covering map $\tilde{t} \mapsto t$. (See Tables 1 and 2.) Such transformations, viewed as taking ${}_2F_1$'s to ${}_2F_1$'s with changed parameters, originated with Gauss and Kummer. The quadratic, cubic, quartic, and sextic ones are classical: they were worked out in detail by Goursat [28]. (Post-19th century expositions are few; Ref. [57] is worth consulting.) A final collection of ${}_2F_1$ transformations, nonclassical in having no free parameter, was discovered only recently [64, 65].

The existence of solution-preserving maps between certain gDH systems, including DH systems, explains the DH representations of Chazy-III solutions recently found by Chakravarty and Ablowitz [13]. Any Chazy-III solution $u = u(\tau)$ can be viewed as the first component of a solution $x = x(\tau)$ of the system $\text{DH}(0, \frac{1}{3}, \frac{1}{2})$, and by lifting this system to a system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, one can generate additional DH representations. For instance, the lifting (3.1) yields $u = (\tilde{x}_1 + \tilde{x}_3)/2$, where $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ is the system $\text{DH}(0, \frac{2}{3}, 0)$. This is one of their representations. In Part II, besides extending their representations to Chazy-I, II, VII, XI, and XII, we shall show that the recently discovered nonclassical transformations of ${}_2F_1$ yield novel and beautiful DH representations of the solutions of the $N = 7, 8, 9$ cases of Chazy-XII.

In principle, a 3-dimensional HQDS $\dot{x} = Q(x)$ such as a gDH system may have a solution-preserving *self-map*: a map that transforms among solutions. A family of self-maps will be generated by any polynomial vector field $q: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ in $\mathcal{C}(Q)$, the centralizer of Q ; i.e., by any vector field $\sum_{i=1}^3 q_i(x) \partial_{x_i}$ in the Lie symmetry algebra of the gDH system. It is known that if the non-associative algebra \mathfrak{A} coming from Q is unital, with a multiplicative identity, then q will be either linear in x , in which case it generates automorphisms of \mathfrak{A} , or quadratic; and the possible quadratic q are bijective with the elements of a certain Jordan sub-algebra of \mathfrak{A} . (See [43] and [66, Chaps. 9,10].) This abstract result is made relevant by being combined with the following: the algebra \mathfrak{A} associated to any DH system is unital, and in fact (see Ohyaama [56]), any 3-dimensional non-associative algebra over \mathbb{C} that is unital is generically isomorphic to one coming from a DH system. It may therefore be possible to classify all solution-preserving self-maps of DH systems.² But here, we focus largely on the task of finding solution-preserving maps Φ between *non-isomorphic* gDH systems, which maps are required to be rational. (In §3.5, we do find an order-48 Coxeter group of isomorphisms between distinct gDH systems.)

Among our rational Φ , the ‘classical’ ones are essentially the maps of Harnad and McKay [32], to whom we owe much. But the context is different. Their maps Φ were between HQDS's satisfied by 3-vectors of quasi-modular forms. The HQDS's were specific DH systems, with no free parameters. Our transformations apply more broadly: the HQDS's mapped between are gDH systems with free parameters. We nonetheless share with Harnad–McKay a focus on the automorphism group of the ‘upper,’ or lifted algebra $\tilde{\mathfrak{A}}$. If the associated ring of polynomial invariants in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ has a finite set of generators χ_1, \dots, χ_r , it is natural to express $x = \Phi(\tilde{x})$ in terms of them. (An example is the quadratic map (3.1a): the only non-trivial automorphism of the system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, which satisfies $\tilde{a}_3 = \tilde{a}_1$ and $\tilde{b}_3 = \tilde{b}_1$ by assumption, interchanges \tilde{x}_1, \tilde{x}_3 ; and $\tilde{x}_2, \tilde{x}_1 + \tilde{x}_3, \tilde{x}_1\tilde{x}_3$ can be chosen as the generators.) In general $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \mapsto (\chi_1, \dots, \chi_r)$ will be solution-preserving from

²The Lie symmetries of non-DH HQDS's may include polynomial vector fields of degree > 2 . For Lotka–Volterra HQDS's, several examples occur in the tables of Ref. [3].

the HQDS $(\mathbb{C}^3, \dot{x} = \tilde{Q}(\tilde{x}))$ to a dynamical system on \mathbb{C}^r . (See [42].) The striking thing is that if one chooses x_1, x_2, x_3 to be appropriate rational functions of the generators, this system will become $(\mathbb{C}^3, \dot{x} = Q(x))$, i.e., another HQDS.

Interestingly, the quadratic map (3.1a), denoted by **2** below, and also the cyclic cubic map **3c**, are solution-preserving even when applied to many HQDS's that are not of the gDH form, and are not linearly equivalent to any gDH system. (See Theorems 3.3 and 3.4.)

3.2. Liftings of PE's. By employing the calculus of Riemann P-symbols, one can derive and explain many identities satisfied by the Gauss function ${}_2F_1$. Each such identity (a ${}_2F_1$ transformation) comes from lifting a Gauss hypergeometric equation (GHE) to another GHE, or more generally a PE to another PE, along a rational covering $R: \mathbb{P}_t^1 \rightarrow \mathbb{P}_{\tilde{t}}^1$. Usually R has an additional interpretation, from the theory of conformal mapping: it maps between Schwarzian triangle functions [54].

The quadratic transformations of ${}_2F_1$, which are especially well known [4], arise as follows. Consider a PE (2.1b) on \mathbb{P}_t^1 , for simplicity with singular points t_1, t_2, t_3 equal to $0, 1, \infty$. Consider the degree-2 map

$$(3.3) \quad t = R(\tilde{t}) = 4\tilde{t}/(1 + \tilde{t})^2,$$

which takes $\tilde{t} = 0, \infty$ to $t = 0$; and also $\tilde{t} = 1$ to $t = 1$ and $\tilde{t} = -1$ to $t = \infty$, both with multiplicity 2. In terms of the characteristic exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$ and their differences $\alpha := \mu'_1 - \mu_1$, $\beta := \mu'_2 - \mu_2$, and $\gamma := \mu'_3 - \mu_3$, one can write

$$(3.4) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_1 + \alpha & \mu_2 + \beta & \mu_3 + \gamma \end{array} \right\} (t) = \left\{ \begin{array}{ccc} 0 & 1 & -1 & \infty \\ \mu_1 & 2\mu_2 & 2\mu_3 & \mu_1 \\ \mu_1 + \alpha & 2\mu_2 + 2\beta & 2\mu_3 + 2\gamma & \mu_1 + \alpha \end{array} \right\} (\tilde{t}).$$

Specializing to the case $\mu_3 = 0$, $\gamma = \frac{1}{2}$ yields

$$(3.5) \quad \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \mu_1 & \mu_2 & 0 \\ \mu_1 + \alpha & \mu_2 + \beta & \frac{1}{2} \end{array} \right\} (t) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \mu_1 & 2\mu_2 & \mu_1 \\ \mu_1 + \alpha & 2\mu_2 + 2\beta & \mu_1 + \alpha \end{array} \right\} (\tilde{t}),$$

because any lifted singular point with exponents $0, 1$ (e.g., $\tilde{t} = -1$ here) is no singular point at all, but rather an ordinary point.

Equation (3.5) describes the lifting of the PE on \mathbb{P}_t^1 to one on $\mathbb{P}_{\tilde{t}}^1$, under the change of variables $t = R(\tilde{t})$. The singular points $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3$ on $\mathbb{P}_{\tilde{t}}^1$ are equal to $0, 1, \infty$, like those on \mathbb{P}_t^1 . The lifting acts on exponent differences by $(\alpha, \beta, \frac{1}{2}) \leftarrow (\alpha, 2\beta, \alpha)$, which is a parametric restriction. In solution terms, the meaning of (3.5) is that if $f(t)$ is a local solution of the original PE, $\tilde{f}(\tilde{t}) := f(R(\tilde{t}))$ will be one of the lifted PE. By setting $\mu_1 = 0$ and comparing (3.5) with the GHE P-symbol (2.2a) and the definition (2.4) of the ${}_2F_1$ parameters $(a, b; c)$, one readily derives

$$(3.6) \quad (1 - t)^{-\mu_2} {}_2F_1(\mu_2, \mu_2 + \frac{1}{2}; 1 - \alpha; t) = (1 - \tilde{t})^{-2\mu_2} {}_2F_1(2\mu_2, 2\mu_2 + \alpha; 1 - \alpha; \tilde{t}).$$

Here the left and right prefactors serve to shift one of the exponents at each of $t = 1$ and $\tilde{t} = 1$ to zero, in agreement with (2.2a).

Equation (3.6) is one of many quadratic transformations of ${}_2F_1$, each with two free parameters. The others can be generated by permuting the singular points $\tilde{t} = 0, 1, \infty$ and/or $t = 0, 1, \infty$, i.e., by pre- and post-composing the covering map R with Möbius transformations that permute them. Each modified R will yield a PE

lifting, but to yield a transformation of ${}_2F_1$ as well, $R(0) = 0$ must be satisfied. This is because ${}_2F_1$ is only defined near the origin, as the sum of the series (2.3).

The features of the PE lifting induced by the canonical degree-2 map R of (3.3), which will be denoted by **2**, are indicated in its *branching schema*, which is $1 + 1 = 2 = 2$. The three ‘slots,’ separated by equals signs, indicate the multiplicities with which points on \mathbb{P}_t^1 are mapped to $t = 0, 1, \infty$ respectively; so a value greater than 1 indicates a critical point of the map. The italicization of the second ‘2’ means that in the PE lifting, the corresponding critical point ($\tilde{t} = -1$) is kept from being a singular point by restricting the exponents at the critical value ‘beneath’ it ($t = 1$) to be $0, \frac{1}{2}$. In the schema of any map used for lifting a PE to a PE, there will be exactly three non-italicized multiplicities, pertaining to $\tilde{t} = 0, 1, \infty$ (in some order).

Another covering yielding a PE lifting is the degree-4 map

$$(3.7) \quad t = R(\tilde{t}) = \frac{16\tilde{t}(1-\tilde{t})}{(1+4\tilde{t}-4\tilde{t}^2)^2} = 1 - \frac{(1-2\tilde{t})^4}{(1+4\tilde{t}-4\tilde{t}^2)^2},$$

which will be denoted by **4_{bq}**. The associated branching schema is $1 + 1 + 2 = 4 = 2(2)$, meaning $1 + 1 + 2 = 4 = 2 + 2$, since (for instance) this R takes $\tilde{t} = 0, 1, \infty$ to $t = 0$, with respective multiplicities $1, 1, 2$; and $\tilde{t} = \frac{1}{2}$ to $t = 1$ with multiplicity 4. By examination, the associated lifting acts as $(\alpha, \frac{1}{4}, \frac{1}{2}) \leftarrow (\alpha, \alpha, 2\alpha)$ on the vector of constrained exponent differences. The resulting quartic ${}_2F_1$ transformation is left as an exercise to the reader. The ‘**bq**’ stands for *biquadratic*, as this quartic covering turns out to factor into two quadratic ones. This will be indicated by **4_{bq}** \sim **2** \circ **2**. The symbol \sim is used because this is not a composition of the above map **2** with itself, a permutation of singular points being involved. Specifically,

$$(3.8) \quad \frac{16\tilde{t}(1-\tilde{t})}{(1+4\tilde{t}-4\tilde{t}^2)^2} = \frac{4\tilde{t}}{(1+\tilde{t})^2} \circ \frac{\tilde{t}}{\tilde{t}-1} \circ \frac{4\tilde{t}}{(1+\tilde{t})^2} \circ \frac{\tilde{t}}{\tilde{t}-1}.$$

In symbolic terms, **4_{bq}** = **2** \circ $\sigma_{(23)}$ \circ **2** \circ $\sigma_{(23)}$, in which each Möbius transformation $\tilde{t} \mapsto \sigma_{(23)}(\tilde{t}) := \tilde{t}/(\tilde{t}-1)$ interchanges the points $1, \infty$.

The quadratic, cubic, quartic and sextic PE liftings (and the resulting ${}_2F_1$ transformations, each with at least one free parameter), were worked out in 1881 by Goursat [28], and the ones yielding transformations with no free parameter much more recently [64, 65]. In Tables 1 and 2 a canonical covering map for each type of ${}_2F_1$ transformation is given, chosen in an *ad hoc* way (there is no generally accepted convention for choosing a representative unique up to isomorphism). Each map $\tilde{t} \mapsto t = R(\tilde{t})$ is written as $-P_1(\tilde{t})/P_3(\tilde{t})$, a quotient of polynomials, so that

$$(3.9) \quad t = -P_1/P_3, \quad (t-1)/t = -P_2/P_1, \quad 1/(1-t) = -P_3/P_2,$$

where $P_1 + P_2 + P_3 = 0$. According to our convention each canonical R must map $\tilde{t} = 0, \infty$ to $t = 0$; also $R^{-1}(\infty)$ must contain at least one ordinary point, so the third ($t = \infty$) slot in the branching schema must include at least one italicized integer > 1 . (All italicized integers in any single slot must be equal.) These requirements are admittedly arbitrary, but loosening them will merely permute the points $\tilde{t} = 0, 1, \infty$ and/or $t = 0, 1, \infty$, and modify P_1, P_2, P_3 correspondingly.

The tabulated maps split neatly into four classes. The ones in the first and second, denoted by **2**, **3**, **4**, **6**, **6_c**, resp. **3_c**, **4_{bq}**, were known to Goursat. The ones in the first can lift PE’s with exponent differences $(\alpha_1, \alpha_2, \alpha_3)$ equal to $(0, \frac{1}{3}, \frac{1}{2})$, so they arise naturally in the context of modular forms. (See Part II; also [49].) The maps **4**, **4_{bq}** are distinct, as are **3**, **3_c** and **6**, **6_c**. The cubic map **3_c** is defined not over \mathbb{Q}

TABLE 1. Standardized covering maps $t = -P_1(\tilde{t})/P_3(\tilde{t})$ appearing in the classical and semiclassical transformations of ${}_2F_1$. Each triple P_1, P_2, P_3 satisfies $P_1 + P_2 + P_3 = 0$. In the $\mathbf{4}_{\mathbf{bq}}, \mathbf{12}_{\mathbf{bq}}$ maps, r signifies $\tilde{t} - 1/2$, and in the $\mathbf{3}_{\mathbf{c}}, \mathbf{6}_{\mathbf{c}}, \mathbf{12}_{\mathbf{c}}$ maps, s, \bar{s} signify $\tilde{t} + \omega, \tilde{t} + \bar{\omega}$, for ω a primitive cube root of unity.

covering map	branching schema	$(\alpha_1, \alpha_2, \alpha_3) \leftarrow (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$	$[P_1, P_2, P_3](\tilde{t})$
2	$1 + 1 = 2 = 2$	$(\alpha, \beta, \frac{1}{2}) \leftarrow (\alpha, 2\beta, \alpha)$	$[-4t, -(1-t)^2, (1+t)^2]$
3	$1 + 2 = 3 = 1 + 2$	$(\alpha, \frac{1}{3}, \frac{1}{2}) \leftarrow (\alpha, \frac{1}{2}, 2\alpha)$	$[-27t, -(1-4t)^3, (1-t)(1+8t)^2]$
4	$1 + 3 = 1 + 3 = (2)2$	$(\alpha, \frac{1}{3}, \frac{1}{2}) \leftarrow (\alpha, \frac{1}{3}, 3\alpha)$	$[-64t, -(1-t)(1-9t)^3, (1+18t-27t^2)^2]$
6 \sim 3 \circ 2	$1 + 1 + 4 = (2)3 = (3)2$	$(\alpha, \frac{1}{3}, \frac{1}{2}) \leftarrow (\alpha, \alpha, 4\alpha)$	$[-108t(1-t), -(1-16t+16t^2)^3, (1-2\tilde{t})^2(1+32\tilde{t}-32\tilde{t}^2)^2]$
6_c \sim 3 \circ 2 \sim 2 \circ 3_c	$2 + 2 + 2 = (2)3 = (3)2$	$(\alpha, \frac{1}{3}, \frac{1}{2}) \leftarrow (2\alpha, 2\alpha, 2\alpha)$	$[27t^2(1-t)^2, -4(1-t+t^2)^3, (1+\tilde{t})^2(2-\tilde{t})^2(1-2\tilde{t})^2]$ $= [- (s^3 - \bar{s}^3)^2, -4s^3\bar{s}^3, (s^3 + \bar{s}^3)^2]$
3_c	$1 + 1 + 1 = 3 = 3$	$(\alpha, \frac{1}{3}, \frac{1}{3}) \leftarrow (\alpha, \alpha, \alpha)$	$[3(\omega - \bar{\omega})t(1-t), -(t+\bar{\omega})^3, (t+\omega)^3]$ $= [\bar{s}^3 - s^3, -\bar{s}^3, s^3]$
4_{bq} \sim 2 \circ 2	$1 + 1 + 2 = 4 = (2)2$	$(\alpha, \frac{1}{4}, \frac{1}{2}) \leftarrow (\alpha, \alpha, 2\alpha)$	$[-16t(1-t), -(1-2t)^4, (1+4\tilde{t}-4\tilde{t}^2)^2]$ $= [-4(1-4r^2), -16r^4, 4(1-2r^2)^2]$
12_{bq} \sim 6 \circ 2 \sim 3 \circ 2 \circ 2 \sim 3 \circ 4_{bq}	$1 + 1 + 2 + 8 = (4)3 = (6)2$	$(\frac{1}{8}, \frac{1}{3}, \frac{1}{2}) \leftarrow (\frac{1}{8}, \frac{1}{8}, \frac{1}{4})$	$[27r^8(1-4r^2), -4(1-4r^2+r^4)^3, (1-2r^2)^2(2-8r^2-r^4)^2]$
12_c \sim 4 \circ 3_c	$1 + 1 + 1 + 9 = (4)3 = (6)2$	$(\frac{1}{9}, \frac{1}{3}, \frac{1}{2}) \leftarrow (\frac{1}{9}, \frac{1}{9}, \frac{1}{9})$	$[64s^9(\bar{s}^3 - s^3), -\bar{s}^3(9\bar{s}^3 - 8s^3)^3, (27\bar{s}^6 - 36s^3\bar{s}^3 + 8s^6)^2]$

TABLE 2. Standardized covering maps $t = -P_1(\tilde{t})/P_3(\tilde{t})$ appearing in the nonclassical transformations of ${}_2F_1$ that are defined over \mathbb{Q} . Each triple P_1, P_2, P_3 satisfies $P_1 + P_2 + P_3 = 0$.

covering map	branching schema, $(\alpha_1, \alpha_2, \alpha_3) \leftarrow (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$	$[P_1, P_2, P_3](\tilde{t})$
10	$1 + 2 + 7$ $= 1 + 3(3) = 5(2)$ $(\frac{1}{7}, \frac{1}{3}, \frac{1}{2}) \leftarrow (\frac{1}{7}, \frac{1}{3}, \frac{2}{7})$	$[t(32 - 81\tilde{t})^7,$ $-4(1 - \tilde{t})(256 + 17280\tilde{t} + 5832\tilde{t}^2 - 6561\tilde{t}^3)^3,$ $(8192 - 1271808\tilde{t} - 6127488\tilde{t}^2 + 7453296\tilde{t}^3 - 1948617\tilde{t}^4 + 1062882\tilde{t}^5)^2]$
24_c \sim 8 \circ 3_c	$1 + 1 + 1 + 3(7)$ $= 8(3) = 12(2)$ $(\frac{1}{7}, \frac{1}{3}, \frac{1}{2}) \leftarrow (\frac{1}{7}, \frac{1}{7}, \frac{1}{7})$	$[-1728t(1 - \tilde{t})(1 + 5\tilde{t} - 8\tilde{t}^2 + \tilde{t}^3)^7,$ $-(1 - \tilde{t} + \tilde{t}^2)^3(1 - 235\tilde{t} + 1430\tilde{t}^2 - 1695\tilde{t}^3 + 270\tilde{t}^4 + 229\tilde{t}^5 + \tilde{t}^6)^3,$ $(1 + 510\tilde{t} - 14631\tilde{t}^2 + 80090\tilde{t}^3 - 218058\tilde{t}^4 + 316290\tilde{t}^5 - 253239\tilde{t}^6$ $+ 131562\tilde{t}^7 - 70998\tilde{t}^8 + 37950\tilde{t}^9 - 8955\tilde{t}^{10} - 522\tilde{t}^{11} + \tilde{t}^{12})^2]$

but over $\mathbb{Q}(\omega)$, where ω is a primitive cube root of unity. The subscript in $\mathbf{3_c}$ stands for ‘cyclic,’ and is used because up to composition with Möbius transformations, $\mathbf{3_c}$ is equivalent to the cyclic covering $\tilde{t} \mapsto \tilde{t}^3$. The map $\mathbf{6_c}$ is not cyclic but merits the **c**, because unlike the other sextic map $\mathbf{6}$, it has $\mathbf{3_c}$ as a right factor.

The degree-12 maps in the third class, denoted by $\mathbf{12_{bq}}$ and $\mathbf{12_c}$ in Table 1, could be called ‘semiclassical’: they are compositions of maps that appear in Goursat’s transformations of ${}_2F_1$, and in consequence appear in certain degree-12 ${}_2F_1$ transformations (which turn out to have no free parameter). But the transformations seem not to have appeared in the literature before [64, 65]. The subscripts on $\mathbf{12_{bq}}$ and $\mathbf{12_c}$ indicate the respective presence of $\mathbf{4_{bq}}$ and $\mathbf{3_c}$ as right factors. It may be that $\mathbf{12_{bq}}$ and $\mathbf{12_c}$ were missed because the transformations of ${}_2F_1$ based on $\mathbf{4_{bq}}$ and $\mathbf{3_c}$ (unlike, say, those based on $\mathbf{2}$ and $\mathbf{3}$) are rather obscure, though they can be found in [28].

The maps in the fourth class, denoted by $\mathbf{10}$ and $\mathbf{24_c}$ in Table 2, appear in the remaining ${}_2F_1$ transformations with no free parameter; or more accurately, in the only two that are defined over \mathbb{Q} . There are five additional ones, the maps in which could be denoted by $\mathbf{6'}, \mathbf{8}, \mathbf{9}, \mathbf{10'}, \mathbf{18}$, after their degrees [64, 65]; but being rather exotic, they will not be considered here in any detail. (The map $\mathbf{8}$ is defined over $\mathbb{Q}(\omega)$, but the others are defined over such algebraic number fields as $\mathbb{Q}(i)$.) The subscript in $\mathbf{24_c}$ serves to indicate the presence of a $\mathbf{3_c}$ right factor, and the consequent cyclic symmetry. Of the four ‘c’ maps, $\mathbf{24_c}$ like $\mathbf{6_c}$ is defined over \mathbb{Q} , though $\mathbf{12_c}$ like $\mathbf{3_c}$ is only defined over $\mathbb{Q}(\omega)$.

In a certain sense, the most symmetrical of the eleven canonical maps in Tables 1 and 2 are $\mathbf{2}$, $\mathbf{3_c}$, and $\mathbf{6_c}$. This follows by considering $\mathbb{C}(\tilde{t}), \mathbb{C}(t)$, the fields of rational functions on $\mathbb{P}_t^1, \mathbb{P}_t^1$. For each of $\mathbf{2}$, $\mathbf{3_c}$, and $\mathbf{6_c}$, the field extension $\mathbb{C}(\tilde{t})/\mathbb{C}(t)$ is Galois, the Galois group $G = G(\mathbb{C}(\tilde{t})/\mathbb{C}(t))$ (i.e., the group of automorphisms of $\mathbb{C}(\tilde{t})$ that fix $\mathbb{C}(t)$) being isomorphic respectively to the cyclic groups $\mathfrak{Z}_2, \mathfrak{Z}_3$, and the symmetric group \mathfrak{S}_3 on three letters. The last of these comprises all Möbius transformations of \mathbb{P}_t^1 that permute $\tilde{t} = 0, 1, \infty$, and the first two are the cyclic subgroups $\tilde{t} \mapsto \{\tilde{t}, 1/\tilde{t}\}$ and $\tilde{t} \mapsto \{\tilde{t}, 1/(1-\tilde{t}), (\tilde{t}-1)/\tilde{t}\}$.

The field extensions $\mathbb{C}(\tilde{t})/\mathbb{C}(t)$ coming from most of the other eight maps in Tables 1 and 2 are non-Galois: for each, some element of $\mathbb{C}(\tilde{t})$ not in $\mathbb{C}(t)$ is fixed by all automorphisms of $\mathbb{C}(\tilde{t})$ that fix $\mathbb{C}(t)$. It can be shown (cf. [32]) that the extensions coming from the maps $\mathbf{3}, \mathbf{4}, \mathbf{6}$, though non-Galois, have the symbolic representations $\mathfrak{S}_3/\mathfrak{Z}_2, \mathfrak{A}_4/\mathfrak{Z}_3, \mathfrak{S}_4/\mathfrak{Z}_4$. Here, representing $\mathbb{C}(\tilde{t})/\mathbb{C}(t)$ by G_1/G_2 indicates that $\mathbb{C}(\tilde{t}), \mathbb{C}(t)$ have a common extension $\mathbb{C}(\tilde{\tilde{t}})$, with $\mathbb{C}(\tilde{\tilde{t}})/\mathbb{C}(t)$ and $\mathbb{C}(\tilde{\tilde{t}})/\mathbb{C}(\tilde{t})$ having respective Galois groups G_1, G_2 . But in each non-Galois case, with G_2 not normal in G_1 , there is no naturally associated group of Möbius transformations of \mathbb{P}_t^1 .

It must be mentioned that each map $\tilde{t} = R(t)$ of Table 1 or 2 that is defined over \mathbb{Q} , except for $\mathbf{10}$, has a conformal mapping interpretation along the lines indicated in Ref. [54]. Let $\tau = \tau(t)$ be a (multivalued) Schwarzian triangle function, each branch of which takes the upper-half t -plane to a hyperbolic triangle in the τ -plane with angles $\pi(\alpha_1, \alpha_2, \alpha_3)$, the vertices being the images of $(0, 1, \infty)$. Then if $\deg R = d$, the multivalued function $\tilde{\tau} = \tilde{\tau}(\tilde{t}) := \tau(R(\tilde{t}))$ will act similarly on the upper-half \tilde{t} -plane, each image triangle in the $\tilde{\tau}$ -plane having angles $\pi(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ and being naturally partitioned into d sub-triangles with angles $\pi(\alpha_1, \alpha_2, \alpha_3)$.

The map **2** is an example. It yields a partition of a hyperbolic triangle with angles $\pi(\alpha, 2\beta, \alpha)$ into two triangles with angles $\pi(\alpha, \beta, \frac{1}{2})$; cf. [54, Fig. 43]. Some discussion of the partitions produced by the remaining maps can be found in Refs. [34, 35]. For instance, the map **24_c** yields an elegant dissection of an equilateral hyperbolic triangle with angles $\pi(\frac{1}{7}, \frac{1}{7}, \frac{1}{7})$ into 24 triangles with angles $\pi(\frac{1}{7}, \frac{1}{3}, \frac{1}{2})$.

3.3. Maps between gDH systems. With the aid of Theorem 2.1, a rich collection of solution-preserving maps between gDH systems can now be derived.

Theorem 3.1. *To each classical PE-lifting covering map $t = R(\tilde{t})$ (of degree d) listed in Table 1, i.e., to each of **2, 3, 4, 6, 6_c** and **3_c, 4_{bq}**, there is associated a rational map $x = \Phi(\tilde{x})$ from a parametrized gDH system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ to a parametrized gDH system $\dot{x} = Q(x)$. It can be viewed as a map $\Phi: \mathbb{P}_{\tilde{x}}^2 \rightarrow \mathbb{P}_x^2$, and is given by*

$$\begin{aligned} x_1 &= \Sigma_1/d, \\ x_1 - x_2 &= -\Sigma_3/d \Sigma_2, \\ x_2 - x_3 &= -\Sigma_6/d \Sigma_2 \Sigma_3, \\ x_3 - x_1 &= -\Sigma_2^2/d \Sigma_3, \end{aligned}$$

where Σ_k , $k = 1, 2, 3, 6$, are certain homogeneous polynomials in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ of degree k , which are listed in Table 3 and satisfy $\Sigma_2^3 + \Sigma_3^2 + \Sigma_6 = 0$. A sufficient condition for Φ to be solution-preserving from $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, i.e., $\text{gDH}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$, to $\dot{x} = Q(x)$, i.e., to $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, is that the respective parameter vectors be restricted and related as specified in Table 4.

Remark. The seemingly *ad hoc* convention for standardizing the maps R of Tables 1 and 2 was chosen to regularize the resulting formulas for the rational functions x_i of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. Choosing another would permute $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ and/or x_1, x_2, x_3 .

Proof. Consider (as it turns out, without loss of generality) the case when each gDH system is proper in the sense of Definition 2.2, and the solutions being mapped between, $x = x(\tau)$, $\tilde{x} = \tilde{x}(\tilde{\tau})$ with $\tau = \tilde{\tau}$, are noncoincident in the sense of Definition 2.4. Then the systems and their solutions will come via Theorem 2.1 from a pair of PE's with solutions $f(t), \tilde{f}(\tilde{t})$. Their singular points $(t_1, t_2, t_3), (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$ will both be taken to be $(0, 1, \infty)$, and the exponent offset vectors $\kappa, \tilde{\kappa}$ to be $(0, 0, 1)$.

The map $x = \Phi(\tilde{x})$, and the restrictions on parameters that must be imposed for it to be solution-preserving, will be derived from the requirement that t, \tilde{t} be related as in § 3.2 by the covering $t = R(\tilde{t}) = -P_1(\tilde{t})/P_3(\tilde{t})$. For the duration, let i, j, k be any cyclic permutation of 1, 2, 3, with $y_k := x_i - x_j$, $\tilde{y}_k := \tilde{x}_i - \tilde{x}_j$.

For each map R in Table 1, the corresponding polynomials $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_6$ in Table 3 will come from a formula for the solution-preserving map $x = \Phi(\tilde{x})$,

$$(3.10) \quad x_i = \tilde{x}_3 + d^{-1} \tilde{y}_3 \left[\tilde{t}(1 - \tilde{t})(P'_i/P_i)(\tilde{t}) \right] \Big|_{\tilde{t} = -\tilde{y}_1/\tilde{y}_3},$$

where $P_1 + P_2 + P_3 = 0$ as usual. In fact, $\Sigma_2, \Sigma_3, \Sigma_6$ will come from

$$(3.11) \quad y_k = d^{-1} \tilde{y}_3 \left\{ \tilde{t}(1 - \tilde{t}) \left[(P'_i/P_i) - (P'_j/P_j) \right] (\tilde{t}) \right\} \Big|_{\tilde{t} = -\tilde{y}_1/\tilde{y}_3},$$

a corollary of (3.10) that will be derived first. The proof is long and is therefore divided into three parts: (I) the derivation of the parameter relationships of Table 4, (II) the derivation of (3.11), and (III) the derivation of (3.10). Of the three parts the first is the longest. Only in (III) will the facts $\kappa, \tilde{\kappa} = (0, 0, 1)$ be used.

TABLE 3. Polynomials Σ_k in the gDH transformations $x = \Phi(\tilde{x})$ derived from classical ${}_2F_1$ transformations. Each triple $\Sigma_2, \Sigma_3, \Sigma_6$ satisfies $\Sigma_2^3 + \Sigma_3^2 + \Sigma_6 = 0$. In **3_c** and **6_c**, z, \bar{z} signify $\tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3$, $\tilde{x}_1 + \bar{\omega}\tilde{x}_2 + \omega\tilde{x}_3$.

covering map	Σ_1	$\Sigma_2, \Sigma_3, \Sigma_6$
2	$\tilde{x}_1 + \tilde{x}_3$	$-(\tilde{x}_3 - \tilde{x}_1)^2,$ $(\tilde{x}_3 - \tilde{x}_1)^2(\tilde{x}_1 - 2\tilde{x}_2 + \tilde{x}_3),$ $4(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1)^4$
3	$\tilde{x}_1 + 2\tilde{x}_3$	$-(\tilde{x}_3 - \tilde{x}_1)(-\tilde{x}_1 - 3\tilde{x}_2 + 4\tilde{x}_3),$ $(\tilde{x}_3 - \tilde{x}_1)^2(\tilde{x}_1 - 9\tilde{x}_2 + 8\tilde{x}_3),$ $-27(\tilde{x}_1 - \tilde{x}_2)^2(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1)^3$
4	$\tilde{x}_1 + 3\tilde{x}_3$	$-(\tilde{x}_3 - \tilde{x}_1)(-\tilde{x}_1 - 8\tilde{x}_2 + 9\tilde{x}_3),$ $(\tilde{x}_3 - \tilde{x}_1)(-\tilde{x}_1^2 + 8\tilde{x}_2^2 + 27\tilde{x}_3^2 + 20\tilde{x}_1\tilde{x}_2 - 36\tilde{x}_2\tilde{x}_3 - 18\tilde{x}_3\tilde{x}_1),$ $64(\tilde{x}_1 - \tilde{x}_2)^3(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1)^2$
6 \sim 3 \circ 2	$\tilde{x}_1 + \tilde{x}_2 + 4\tilde{x}_3$	$-(\tilde{x}_1^2 + \tilde{x}_2^2 + 16\tilde{x}_3^2 + 14\tilde{x}_1\tilde{x}_2 - 16\tilde{x}_2\tilde{x}_3 - 16\tilde{x}_3\tilde{x}_1),$ $(\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3)(\tilde{x}_1^2 + \tilde{x}_2^2 - 32\tilde{x}_3^2 - 34\tilde{x}_1\tilde{x}_2 + 32\tilde{x}_2\tilde{x}_3 + 32\tilde{x}_3\tilde{x}_1),$ $-108(\tilde{x}_1 - \tilde{x}_2)^4(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1)$
6_c \sim 3 \circ 2 \sim 2 \circ 3_c	$2\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3$	$-4(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 - \tilde{x}_1\tilde{x}_2 - \tilde{x}_2\tilde{x}_3 - \tilde{x}_3\tilde{x}_1) = -4z\bar{z},$ $4(2\tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3)(2\tilde{x}_2 - \tilde{x}_3 - \tilde{x}_1)(2\tilde{x}_3 - \tilde{x}_1 - \tilde{x}_2) = 4(z^3 + \bar{z}^3),$ $432(\tilde{x}_1 - \tilde{x}_2)^2(\tilde{x}_2 - \tilde{x}_3)^2(\tilde{x}_3 - \tilde{x}_1)^2 = -16(\bar{z}^3 - z^3)^2$
3_c	$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$	$-(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 - \tilde{x}_1\tilde{x}_2 - \tilde{x}_2\tilde{x}_3 - \tilde{x}_3\tilde{x}_1) = -(\tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3)(\tilde{x}_1 + \bar{\omega}\tilde{x}_2 + \omega\tilde{x}_3) =: -z\bar{z},$ $(\tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3)^3 =: z^3,$ $3(\omega - \bar{\omega})(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1) \cdot (\tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3)^3 = (\bar{z}^3 - z^3)z^3$
4_{bq} \sim 2 \circ 2	$\tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3$	$-(\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3)^2,$ $(\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3)(\tilde{x}_1^2 + \tilde{x}_2^2 - 4\tilde{x}_3^2 - 6\tilde{x}_1\tilde{x}_2 + 4\tilde{x}_2\tilde{x}_3 + 4\tilde{x}_3\tilde{x}_1),$ $-16(\tilde{x}_1 - \tilde{x}_2)^2(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1) \cdot (\tilde{x}_1 + \tilde{x}_2 - 2\tilde{x}_3)^2$

TABLE 4. Constraints on the parameters of the systems $\text{gDH}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ and $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, under which $x = \Phi(\tilde{x})$ is a solution-preserving map. (In all cases, $c = \tilde{c}$.)

covering map	$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$	$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$	(a_1, a_2, a_3)	(b_1, b_2, b_3)
2	$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_1)$	$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_1)$	$(2\tilde{a}_1, \tilde{a}_2, 2\tilde{a}_1 + \tilde{a}_2 - \tilde{c})$	$(\tilde{c} - \tilde{b}_2, \tilde{b}_2, 2\tilde{b}_1 + \tilde{b}_2 - \tilde{c})$
3	$(\tilde{a}, 3\tilde{a} - \tilde{c}, 2\tilde{a})$	$(\tilde{b}, 3\tilde{b} - \tilde{c}, \tilde{c} - \tilde{b})$	$(3\tilde{a}, 2(3\tilde{a} - \tilde{c}), 3(3\tilde{a} - \tilde{c}))$	$(2\tilde{c} - 3\tilde{b}, 3\tilde{b} - \tilde{c}, 3\tilde{b} - \tilde{c})$
4	$(\tilde{a}, \frac{1}{2}(4\tilde{a} - \tilde{c}), 3\tilde{a})$	$(\tilde{b}, \frac{1}{2}(4\tilde{b} - \tilde{c}), \tilde{c} - \tilde{b})$	$(4\tilde{a}, 2(4\tilde{a} - \tilde{c}), 3(4\tilde{a} - \tilde{c}))$	$(\frac{1}{2}(3\tilde{c} - 4\tilde{b}), \frac{1}{2}(4\tilde{b} - \tilde{c}), \frac{1}{2}(4\tilde{b} - \tilde{c}))$
6 \sim 3 \circ 2	$(\tilde{a}, \tilde{a}, 4\tilde{a})$	$(\tilde{b}, \tilde{b}, \frac{1}{4}(3\tilde{c} - 2\tilde{b}))$	$(6\tilde{a}, 2(6\tilde{a} - \tilde{c}), 3(6\tilde{a} - \tilde{c}))$	$(\frac{1}{4}(5\tilde{c} - 6\tilde{b}), \frac{1}{4}(6\tilde{b} - \tilde{c}), \frac{1}{4}(6\tilde{b} - \tilde{c}))$
6_c \sim 3 \circ 2 \sim 2 \circ 3_c	$(\tilde{a}, \tilde{a}, \tilde{a})$	$(\tilde{b}, \tilde{b}, \tilde{b})$	$(3\tilde{a}, 2(3\tilde{a} - \tilde{c}), 3(3\tilde{a} - \tilde{c}))$	$(2\tilde{c} - 3\tilde{b}, 3\tilde{b} - \tilde{c}, 3\tilde{b} - \tilde{c})$
3_c	$(\tilde{a}, \tilde{a}, \tilde{a})$	$(\tilde{b}, \tilde{b}, \tilde{b})$	$(3\tilde{a}, 3\tilde{a} - \tilde{c}, 3\tilde{a} - \tilde{c})$	$(2\tilde{c} - 3\tilde{b}, 3\tilde{b} - \tilde{c}, 3\tilde{b} - \tilde{c})$
4_{bq} \sim 2 \circ 2	$(\tilde{a}, \tilde{a}, 2\tilde{a})$	$(\tilde{b}, \tilde{b}, \frac{1}{2}\tilde{c})$	$(4\tilde{a}, 4\tilde{a} - \tilde{c}, 2(4\tilde{a} - \tilde{c}))$	$(\frac{1}{2}(3\tilde{c} - 4\tilde{b}), \frac{1}{2}(4\tilde{b} - \tilde{c}), \frac{1}{2}(4\tilde{b} - \tilde{c}))$
12_{bq} \sim 6 \circ 2 \sim 3 \circ 2 \circ 2 \sim 3 \circ 4_{bq}	$(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2})\tilde{c}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\tilde{c}$	$(-3, -8, -12)\tilde{c}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\tilde{c}$
12_c \sim 4 \circ 3_c	$(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})\tilde{c}$	"	$(-2, -6, -9)\tilde{c}$	"
10	$(-\frac{3}{8}, -\frac{7}{8}, -\frac{3}{8})\tilde{c}$	"	$(-6, -14, -21)\tilde{c}$	"
24_c \sim 8 \circ 3_c	$(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})\tilde{c}$	"	$(-6, -14, -21)\tilde{c}$	"

Part I of Proof. What must first be elucidated are the relation between f, \tilde{f} , and that between the exponents of the PE's that f, \tilde{f} satisfy; and hence, the relation between the parameters of the two induced gDH systems. It is not the case that $f(t) = \tilde{f}(\tilde{t})$. (The reader should glance at the identity (3.5), which shows the P-symbols of two PE's related by the quadratic transformation **2**, the solutions f, \tilde{f} of which *do* satisfy $f(t) = \tilde{f}(\tilde{t})$.) That $f(t), \tilde{f}(\tilde{t})$ must differ is evident from the parametrizations $\tau = \tau(t), \tilde{\tau} = \tilde{\tau}(\tilde{t})$ produced by Theorem 2.1, which satisfy

$$(3.12) \quad \frac{d\tau}{dt} = K^{-2}(t)f^{-1/\bar{n}}(t), \quad \frac{d\tilde{\tau}}{d\tilde{t}} = \tilde{K}^{-2}(\tilde{t})\tilde{f}^{-1/\bar{n}}(\tilde{t}),$$

for some chosen $\bar{n} \in \mathbb{P}^1 \setminus \{0, \infty\}$ and functions K, \tilde{K} determined by the chosen offset vectors $\kappa, \tilde{\kappa}$. If $t = R(\tilde{t})$ and $\tau = \tilde{\tau}$ then

$$(3.13) \quad \tilde{f}(\tilde{t})/f(t) = \left[K^2(t)/\tilde{K}^2(\tilde{t})R'(\tilde{t}) \right]^{\bar{n}},$$

the prime indicating differentiation with respect to \tilde{t} . It follows readily from (3.13) that if a point $t^* \in \mathbb{P}_t^1$ is mapped by R with multiplicity m from a point $\tilde{t}^* \in \mathbb{P}_{\tilde{t}}^1$, so that the local behavior of R is $(t - t^*) \sim \text{const} \times (\tilde{t} - \tilde{t}^*)^m$, then the respective exponents (μ, μ') and $(\tilde{\mu}, \tilde{\mu}')$ are related not by $(\tilde{\mu}, \tilde{\mu}') = m(\mu, \mu')$ (as is the case, say, in (3.5)), but rather by

$$(3.14) \quad [(\tilde{\mu}, \tilde{\mu}') + (2\tilde{\kappa} - 1)\bar{n}(1, 1)] = m[(\mu, \mu') + (2\kappa - 1)\bar{n}(1, 1)].$$

Here, $\kappa, \tilde{\kappa}$ are specific to the points t^*, \tilde{t}^* in the sense that κ is κ_i if $t^* = t_i$, and $\tilde{\kappa}$ is $\tilde{\kappa}_i$ if $\tilde{t}^* = \tilde{t}_i$; and otherwise they are zero.

Equation (3.14) takes on a simple form when written in terms of *offset* exponents $\nu := \mu + (2\kappa - 1)\bar{n}$, etc., which were already introduced in (2.15), in the statement of Theorem 2.1. It says that

$$(3.15) \quad (\tilde{\nu}, \tilde{\nu}') = m(\nu, \nu')$$

relates the (offset) exponents at any point $\tilde{t} = \tilde{t}^*$ to those at its image $t = t^*$. Informally, when lifting a PE on \mathbb{P}_t^1 that induces a gDH system (with independent variable τ) to a PE on $\mathbb{P}_{\tilde{t}}^1$ that induces another gDH system (with the *same* independent variable, i.e., with $\tilde{\tau} = \tau$), one should work in terms of ν 's (offset exponents) rather than μ 's (unoffset ones).

An example is the map **2**, i.e., $t = R(\tilde{t}) = 4\tilde{t}/(1 + \tilde{t})^2$. Equation (3.15) holding, the P-symbol identity (3.5) must be replaced by the pair of P-symbols

$$(3.16a) \quad f(t) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \nu_1 - (2\kappa_1 - 1)\bar{n} & \nu_2 - (2\kappa_2 - 1)\bar{n} & 0 - (2\kappa_3 - \frac{1}{2})\bar{n} \\ \nu'_1 - (2\kappa_1 - 1)\bar{n} & \nu'_2 - (2\kappa_2 - 1)\bar{n} & \frac{1}{2} - (2\kappa_3 - \frac{1}{2})\bar{n} \end{array} \right\} (t),$$

$$(3.16b) \quad \tilde{f}(\tilde{t}) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \nu_1 - (2\tilde{\kappa}_1 - 1)\bar{n} & 2\nu_2 - (2\tilde{\kappa}_2 - 1)\bar{n} & \nu_1 - (2\tilde{\kappa}_3 - 1)\bar{n} \\ \nu'_1 - (2\tilde{\kappa}_1 - 1)\bar{n} & 2\nu'_2 - (2\tilde{\kappa}_2 - 1)\bar{n} & \nu'_1 - (2\tilde{\kappa}_3 - 1)\bar{n} \end{array} \right\} (\tilde{t}),$$

which are parametrized by the offset exponents $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$, together with $\kappa, \tilde{\kappa}$ and \bar{n} . Most of the dependencies that one sees between exponents in (3.16a), (3.16b) are attributable to the unitalicized multiplicities in the branching schema $1 + 1 = 2 = 2$ of the map **2**, which imply

$$(3.17) \quad (\tilde{\nu}_1, \tilde{\nu}'_1) = (\nu_1, \nu'_1), \quad (\tilde{\nu}_2, \tilde{\nu}'_2) = 2(\nu_2, \nu'_2), \quad (\tilde{\nu}_3, \tilde{\nu}'_3) = (\nu_1, \nu'_1).$$

In (3.16a), the exponents (μ_3, μ'_3) at $t = t_3 = \infty$, which is doubly mapped from $\tilde{t} = -1$, come instead from the fact that at any point $t^* \in \mathbb{P}_t^1$ mapped with multiplicity m from an *ordinary* point $\tilde{t}^* \in \mathbb{P}_{\tilde{t}}^1$ (with exponents $(\tilde{\mu}, \tilde{\mu}')$ equal to $(0, 1)$ by definition), the offset exponents (ν, ν') and exponents (μ, μ') are given by

$$(3.18a) \quad (\nu, \nu') = (0, 1/m) - (1/m)\bar{n}(1, 1),$$

$$(3.18b) \quad (\mu, \mu') = (0, 1/m) - (2\kappa - 1 + 1/m)\bar{n}(1, 1),$$

due to (3.15). It should be noted that $\nu_1, \nu_2, \nu_3; \nu'_1, \nu'_2, \nu'_3$ are always constrained by the Fuchsian condition

$$(3.19) \quad \sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n},$$

which follows from Fuchs's relation $\sum_{i=1}^3 (\mu_i + \mu'_i) = 1$. In (3.16), each of $\nu_1, \nu'_1; \nu_2, \nu'_2$ can be viewed as a free parameter, \bar{n} being determined by (3.19).

It can now be seen how the gDH parameter vectors $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ and $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ are related, subject to constraints. To begin with, the ‘upper’ offset exponents $(\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3; \tilde{\nu}'_1, \tilde{\nu}'_2, \tilde{\nu}'_3)$ will satisfy one or more constraints. (For instance, for the map **2** these are $\tilde{\nu}_3 = \tilde{\nu}_1$ and $\tilde{\nu}'_3 = \tilde{\nu}'_1$, by (3.17).) Let $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ be computed from $(\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3; \tilde{\nu}'_1, \tilde{\nu}'_2, \tilde{\nu}'_3)$, by the formulas (2.16) of Theorem 2.1. Then $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ will also be constrained; for **2** the constraints turn out to be that $\tilde{a}_3 = \tilde{a}_1$, $\tilde{b}_3 = \tilde{b}_1$. Let the ‘lower’ offset exponents $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$ be computed from $(\tilde{\nu}_1, \tilde{\nu}'_1; \tilde{\nu}_2, \tilde{\nu}'_2; \tilde{\nu}_3, \tilde{\nu}'_3)$ by (3.15), using the unitalicized multiplicities in the branching schema of R , and let $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ be computed from $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$ by the formulas (2.14) of Theorem 2.1. What results is an expression for the gDH parameter vector $(a_1, a_2, a_3; b_1, b_2, b_3; c)$, as a function of the (constrained) gDH parameter vector $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$.

The parametric constraints and relations in Table 4 were computed by this technique. (Each line of the table was normalized by taking $c = \tilde{c}$.) In most cases the constraints on $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ have an interpretation in terms of symmetry. For instance, for each ‘c’ map the constraints are that $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3$ and $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3$, because the singular points $\tilde{t} = 0, 1, \infty$ are mapped by R to the same point ($t = 0$), with equal multiplicities.

As was noted in § 2.2, it is implicit in Theorem 2.1 that the gDH system that it produces is proper. Here, this translates to a hidden assumption that the two gDH systems being mapped between do *not* satisfy (i) $\tilde{c} = 0$ or $c = 0$, (ii) $2\tilde{c} - \tilde{b}_1 - \tilde{b}_2 - \tilde{b}_3 = 0$ or $2c - b_1 - b_2 - b_3 = 0$, or (iii) $\tilde{c} - \tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3 = 0$ or $c - a_1 - a_2 - a_3 = 0$. But by continuity of each gDH system in its parameters, this assumption can safely be dropped.

Part II of Proof. Now that the relation between the two PE's, the relation between their solutions $f(t), \tilde{f}(\tilde{t})$, and that between the two induced gDH systems $\dot{x} = Q(x), \dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ are all understood, the solution-preserving map $x = \Phi(\tilde{x})$ can be computed from the covering $t = R(\tilde{t})$. First, from f, \tilde{f} , construct as functions of t, \tilde{t} the dependent variables x, \tilde{x} and independent variables $\tau, \tilde{\tau}$ (satisfying $\tau = \tilde{\tau}$),

as in Theorem 2.1. Define $\rho, \tilde{\rho}$ by

$$(3.20) \quad \begin{aligned} \rho^{-1} &= \mu_1 + \mu_2 + \mu_3 & \tilde{\rho}^{-1} &= \tilde{\mu}_1 + \tilde{\mu}_2 + \tilde{\mu}_3 \\ &= \frac{2c - b_1 - b_2 - b_3}{c - a_1 - a_2 - a_3}, & &= \frac{2\tilde{c} - \tilde{b}_1 - \tilde{b}_2 - \tilde{b}_3}{\tilde{c} - \tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3}, \end{aligned}$$

the lower expressions following from the formulas in that theorem; and note that $\rho/\tilde{\rho} = d$, as R is a degree- d covering. (The case when ρ or $\tilde{\rho}$ is undefined can be handled by continuity, as above.) As (t_1, t_2, t_3) and $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$ equal $(0, 1, \infty)$, the formula (2.34) of Lemma 2.5 becomes

$$(3.21) \quad t = -\left(\frac{x_2 - x_3}{x_1 - x_2}\right) = -\frac{y_1}{y_3}, \quad \tilde{t} = -\left(\frac{\tilde{x}_2 - \tilde{x}_3}{\tilde{x}_1 - \tilde{x}_2}\right) = -\frac{\tilde{y}_1}{\tilde{y}_3},$$

it being assumed that in the gDH solutions $x = x(\tau)$ and $\tilde{x} = \tilde{x}(\tilde{\tau})$ are noncoincident. (The case of coinciding components can be handled by another continuity argument.) Also, Eq. (2.39) of the lemma yields

$$(3.22) \quad \dot{t} = c\bar{n}\tilde{\rho} \left[\frac{(\tilde{x}_2 - \tilde{x}_3)(\tilde{x}_3 - \tilde{x}_1)}{\tilde{x}_1 - \tilde{x}_2} \right] = c\bar{n}\tilde{\rho} \left(\frac{\tilde{y}_1\tilde{y}_2}{\tilde{y}_3} \right).$$

As $\dot{t} = \dot{\tilde{t}}R'(\tilde{t})$, it then follows from Eq. (2.33) of the lemma that

$$(3.23) \quad \begin{aligned} y_k = x_i - x_j &= c^{-1}\bar{n}^{-1}\rho^{-1} \left[\frac{\dot{t}}{t - t_i} - \frac{\dot{t}}{t - t_j} \right] \\ &= c^{-1}\bar{n}^{-1}\rho^{-1} \left[\frac{1}{t - t_i} - \frac{1}{t - t_j} \right] \dot{t} R'(\tilde{t}) \\ &= d^{-1} \left[\frac{1}{R(\tilde{t}) - t_i} - \frac{1}{R(\tilde{t}) - t_j} \right] \left(\frac{\tilde{y}_1\tilde{y}_2}{\tilde{y}_3} \right) R'(\tilde{t}). \end{aligned}$$

But $R(\tilde{t}) = -P_1(\tilde{t})/P_3(\tilde{t})$ with $P_1 + P_2 + P_3 = 0$, and $(t_1, t_2, t_3) = (0, 1, \infty)$. Using (3.22) and (3.21) to rewrite \dot{t} in terms of \tilde{y}_3 and \tilde{t} , one obtains rational expressions for y_1, y_2, y_3 in terms of $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$, which by examination are

$$(3.24) \quad (y_1, y_2, y_3)/\tilde{y}_3 = d^{-1} \left[\tilde{t}(1 - \tilde{t})R'(\tilde{t}) \cdot (P_2^{-1}P_3, P_1^{-1}P_3, P_1^{-1}P_2^{-1}P_3^2) \right] \Big|_{\tilde{t}=-\tilde{y}_1/\tilde{y}_3}.$$

Further rearrangement using $R' = (P_3'P_1 - P_1'P_3)/P_3^2$ and $P_1' + P_2' + P_3' = 0$ yields

$$(3.25) \quad y_k = d^{-1}\tilde{y}_3 \left\{ \tilde{t}(1 - \tilde{t}) \left[(P_i'/P_i) - (P_j'/P_j) \right] (\tilde{t}) \right\} \Big|_{\tilde{t}=-\tilde{y}_1/\tilde{y}_3}$$

for $k = 1, 2, 3$, which was claimed as Eq. (3.11) above.

The expressions for $\Sigma_2, \Sigma_3, \Sigma_6$ in Table 3 can be computed from Eq. (3.25), which gives each $y_k = x_i - x_j$ as a rational function of $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$, and hence of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. Actually, it is more illuminating to use (3.24). Writing y_1, y_2, y_3 in terms of $\Sigma_2, \Sigma_3, \Sigma_6$, and using (3.24) to solve for the latter, yields

$$(3.26a) \quad \Sigma_2/\tilde{y}_3^2 = \left\{ [\tilde{t}(\tilde{t} - 1)R']^2 P_1^{-2}P_2^{-1}P_3^3 \right\} \Big|_{\tilde{t}=-\tilde{y}_1/\tilde{y}_3},$$

$$(3.26b) \quad \Sigma_3/\tilde{y}_3^3 = \left\{ [\tilde{t}(\tilde{t} - 1)R']^3 P_1^{-3}P_2^{-2}P_3^5 \right\} \Big|_{\tilde{t}=-\tilde{y}_1/\tilde{y}_3},$$

$$(3.26c) \quad \Sigma_6/\tilde{y}_3^6 = \left\{ [\tilde{t}(\tilde{t} - 1)R']^6 P_1^{-5}P_2^{-4}P_3^9 \right\} \Big|_{\tilde{t}=-\tilde{y}_1/\tilde{y}_3}.$$

By considering the order of vanishing of each right side in (3.26) at each zero of P_1, P_2 , or P_3 , it is not difficult to see that irrespective of the choice of classical

PE-lifting map $t = R(\tilde{t})$, each of $\Sigma_2, \Sigma_3, \Sigma_6$ must be *polynomial* in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, and not merely rational. (This is essentially because in each case, $R^{-1}\{0\} \subset \{0, 1, \infty\}$.) Polynomiality also follows by direct computation.

Part III of Proof. What remains to be derived is the strengthening (3.10) of (3.25), i.e., the useful formula

$$(3.27) \quad x_i = \tilde{x}_3 + d^{-1} \tilde{y}_3 [\tilde{t}(1 - \tilde{t})(P'_i/P_i)(\tilde{t})] \big|_{\tilde{t} = -\tilde{y}_1/\tilde{y}_3},$$

from which the remaining polynomials Σ_1 of Table 3 can be computed. Equation (3.27) is a corollary of an even simpler formula,

$$(3.28) \quad f'_i(t = R(\tilde{t}))/\tilde{f}'_3(\tilde{t}) = C \times P_i(\tilde{t}) \big|_{\tilde{t} = -\tilde{y}_1/\tilde{y}_3},$$

where as in Theorem 2.1, $f_i = \Delta_i f$ and $\tilde{f}_i = \tilde{\Delta}_i \tilde{f}$, and where the prefactor $C \neq 0$ is \tilde{t} -independent. To see that (3.27) follows from (3.28), recall (see Theorem 2.1) that x_i, \tilde{x}_i are defined as logarithmic derivatives with respect to τ , i.e.,

$$(3.29) \quad x_i = c^{-1} \bar{n}^{-1} \dot{f}_i/f_i, \quad \tilde{x}_i = c^{-1} \bar{n}^{-1} \dot{\tilde{f}}_i/\tilde{f}_i.$$

By taking the logarithmic derivative of both sides of (3.28), and using (3.22) and (3.21) to rewrite $\dot{\tilde{t}}$ in terms of \tilde{y}_3 and \tilde{t} , one obtains (3.27).

To prove that Eq. (3.28) holds for each classical PE-lifting map, reason as follows. It suffices to prove the $i = 3$ case, i.e., that

$$(3.30) \quad \left[f_3(t)/\tilde{f}_3(\tilde{t}) \right]^\rho = C \times P_3(\tilde{t}).$$

But the relation between f, \tilde{f} is known; see Eq. (3.13) above. The case when $\kappa, \tilde{\kappa} = (0, 0, 1)$ is especially simple, as then $K^2, \tilde{K}^2 = 1$ (see the second remark after Theorem 2.1). The relation becomes $\tilde{f}/f = [R'(\tilde{t})]^{\bar{n}}$, so that

$$(3.31) \quad \begin{aligned} \left[f_3(t)/\tilde{f}_3(\tilde{t}) \right]^\rho &= [R'(\tilde{t})]^{-\bar{n}\rho} \left[\frac{\Delta_3(t)}{\tilde{\Delta}_3(\tilde{t})} \right]^\rho \\ &= [R'(\tilde{t})]^{-\bar{n}\rho} \left[\frac{(-t)^{-\mu_1}(t-1)^{-\mu_2}}{(-\tilde{t})^{-\tilde{\mu}_1}(\tilde{t}-1)^{-\tilde{\mu}_2}} \right]^\rho \\ &= [R'(\tilde{t})]^{-\bar{n}\rho} \left[\frac{(P_1/P_3)^{-\mu_1}(P_2/P_3)^{-\mu_2}}{(-\tilde{t})^{-\tilde{\mu}_1}(\tilde{t}-1)^{-\tilde{\mu}_2}} \right]^\rho. \end{aligned}$$

The expressions for $\Delta_3, \tilde{\Delta}_3$ are taken from (2.18), and use has been made of the fact that $t = R(\tilde{t}) = -P_1(\tilde{t})/P_3(\tilde{t})$ with $P_1 + P_2 + P_3 = 0$. After a bit more manipulation, making use of $\rho^{-1} = \mu_1 + \mu_2 + \mu_3$ (see (3.20) above), one finds that (3.30) will follow from (3.31) if

$$(3.32) \quad \Lambda(\tilde{t}) := (-\tilde{t})^{\tilde{\mu}_1}(\tilde{t}-1)^{\tilde{\mu}_2} P_1^{-\mu_1} P_2^{-\mu_2} P_3^{-\mu_3} [R' = (-P_1/P_3)' = (P_2/P_3)']^{\bar{n}}$$

is a nonzero constant function of \tilde{t} .

To prove this last, it suffices to show that at all \tilde{t} (or merely, all finite \tilde{t}), Λ has zero order of vanishing. But, the only points on $\mathbb{P}^1_{\tilde{t}}$ at which it can have nonzero order of vanishing are those in $R^{-1}(\{0, 1, \infty\})$, i.e., (i) the singular points $\tilde{t} = 0, 1, \infty$, and (ii) ordinary (nonsingular) points mapped by R to $t = 0, 1, \infty$.

(i) As an example, consider $\tilde{t} = 0$, which by convention is mapped by R to $t = 0$. Suppose this is with multiplicity m , i.e., $t = R(\tilde{t}) \sim \text{const} \times \tilde{t}^m$, so that $P_1(\tilde{t}) \sim \text{const} \times \tilde{t}^m$ and $P_2(0), P_3(0) \neq 0$. By (3.32), the order of vanishing of Λ at $\tilde{t} = 0$ will be $\tilde{\mu}_1 - m\mu_1 + (m-1)\bar{n}$. But by the remarks on lifting of exponents

in Part I of this proof, the lifted exponent $\tilde{\mu}_1$ at $\tilde{t} = 0$ will equal $m\mu_1 - (m-1)\bar{n}$. (See (3.14); $\kappa_1 = 0$ and $\tilde{\kappa}_1 = 0$ are used here.) Hence the order of vanishing at $\tilde{t} = 0$ is zero. The singular point $\tilde{t} = 1$ is handled identically.

(ii) As an example, consider $\tilde{t} = \tilde{t}^*$, some ordinary point mapped by R to $t = \infty$. Being an ordinary (nonsingular) point, it has exponents $(\tilde{\mu}, \tilde{\mu}') = (0, 1)$. Suppose the mapping is with multiplicity m , so that $P_3(\tilde{t}) \sim \text{const} \times (\tilde{t} - \tilde{t}^*)^m$ and $P_2(\tilde{t}^*), P_3(\tilde{t}^*) \neq 0$. Then by the above remarks on lifting of exponents, the first exponent μ_3 at the mapped point $t = \infty$ must be $0 - (2 \cdot 1 - 1 + 1/m)\bar{n}$. (See (3.18b); $\kappa_3 = 1$ is used here.) It follows immediately from (3.32) that Λ has zero order of vanishing at $\tilde{t} = \tilde{t}^*$. Ordinary points on \mathbb{P}_t^1 that are mapped by R to the other two singular points, i.e., to $t = 1, \infty$, are handled similarly.

Now that Eq. (3.28) and its corollary (3.27) have been derived, one can readily compute $x_1 =: \Sigma_1/d$ as a function of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ for each classical PE-lifting map $t = R(\tilde{t}) = -(P_1/P_3)(\tilde{t})$. In each case Σ_1 turns out to be linear in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. The resulting linear polynomials are listed in Table 3. \square

Remark. There is an alternative proof of Theorem 3.1, which would go as follows. (I) Each covering $t = R(\tilde{t})$ induces a lifting of a gSE of the form (2.41) satisfied by $t = t(\tau)$ to one for $\tilde{t} = \tilde{t}(\tau)$, and an accompanying lifting of (suitably constrained) parameter vectors, $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}) \leftarrow (\tilde{\nu}_1, \tilde{\nu}'_1; \tilde{\nu}_2, \tilde{\nu}'_2; \tilde{\nu}_3, \tilde{\nu}'_3; \bar{n})$. (II) By Theorem 2.9, each such yields a rational solution-preserving map of (suitably constrained) proper gDH systems. Details are left to the reader.

One should note that for each classical PE-lifting map R , the coefficients of the polynomial Σ_1 in Table 3, which is linear in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, sum to the degree d of the map. In fact the coefficients are the multiplicities with which $\tilde{t} = 0, 1, \infty$ are taken to $t = 0$ by the map (cf. the branching schemata of Table 1).

In the cases **2_c**, **3_c**, **6_c**, in which the field extension $\mathbb{C}(\tilde{t})/\mathbb{C}(t)$ is Galois, there is a simple interpretation of the polynomials $\Sigma_1, \Sigma_2, \Sigma_3$. On the level of the rational solution-preserving map $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \mapsto (x_1, x_2, x_3)$, the respective Galois groups $\mathfrak{Z}_2, \mathfrak{Z}_3, \mathfrak{S}_2$ comprise interchanges $\tilde{x}_1 \leftrightarrow \tilde{x}_3$, cyclic permutations $\tilde{x}_1 \rightarrow \tilde{x}_2 \rightarrow \tilde{x}_3 \rightarrow \tilde{x}_1$, and arbitrary permutations of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. In each case $\Sigma_1, \Sigma_2, \Sigma_3$ form an algebraic basis for an associated ring of polynomial invariants. For instance, in the case **6_c** they generate all symmetric polynomials in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. This interpretation of $\Sigma_1, \Sigma_2, \Sigma_3$ has an extension to the non-Galois cases (see [32]).

The solution-preserving maps between gDH systems induced by the semiclassical and nonclassical PE-lifting maps **12_{bq}**, **12_c** and **10**, **24_c** of Tables 1 and 2 are rather complicated and will not be given in their entirety. The following will suffice.

Theorem 3.2. *To each of **12_{bq}**, **12_c** and **10**, **24_c** (of degree d) there is associated a rational map $x = \Phi(\tilde{x})$ between gDH systems $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ and $\dot{x} = Q(x)$ of the form (1.4). It can be viewed as a map $\Phi: \mathbb{P}_x^2 \rightarrow \mathbb{P}_{\tilde{x}}^2$, and is expressed as in Theorem 3.1 but with $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_6$ rational rather than polynomial, and with*

$$x_1 = \Sigma_1/d = \frac{1}{d} \left[\hat{\Sigma}_1 + \frac{d - d_0}{\deg \Upsilon} \frac{\sum_{i=1}^3 \tilde{x}_j \tilde{x}_k \partial \Upsilon / \partial \tilde{x}_i}{\Upsilon} \right],$$

in which j, k are the elements of $1, 2, 3$ other than i . Here $\hat{\Sigma}_1, \Upsilon$ are certain homogeneous polynomials in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ that are listed in Table 5, the first being linear, and $d_0 < d$ is the sum of the coefficients of $\hat{\Sigma}_1$. A sufficient condition for Φ to

covering map	$\hat{\Sigma}_1$	Υ
$\mathbf{12}_{\mathbf{bq}} \sim \mathbf{6} \circ \mathbf{2}$ $\sim \mathbf{3} \circ \mathbf{2} \circ \mathbf{2}$ $\sim \mathbf{3} \circ \mathbf{4}_{\mathbf{bq}}$	$\tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3$	$\tilde{x}_1 - 2\tilde{x}_2 + \tilde{x}_3$
$\mathbf{12}_{\mathbf{c}} \sim \mathbf{4} \circ \mathbf{3}_{\mathbf{c}}$	$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$	$\tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3 =: z$
$\mathbf{10}$	$\tilde{x}_1 + 2\tilde{x}_3$	$32\tilde{x}_1 + 49\tilde{x}_2 - 81\tilde{x}_3$
$\mathbf{24}_{\mathbf{c}} \sim \mathbf{8} \circ \mathbf{3}_{\mathbf{c}}$	$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$	$(\tilde{x}_1^3 + \tilde{x}_2^3 + \tilde{x}_3^3) + 5(\tilde{x}_1^2\tilde{x}_2 + \tilde{x}_2^2\tilde{x}_3 + \tilde{x}_3^2\tilde{x}_1)$ $- 8(\tilde{x}_1\tilde{x}_2^2 + \tilde{x}_2\tilde{x}_3^2 + \tilde{x}_3\tilde{x}_1^2) + 6\tilde{x}_1\tilde{x}_2\tilde{x}_3$

TABLE 5. Polynomials $\hat{\Sigma}_1, \Upsilon$ in the gDH transformations $x = \Phi(\tilde{x})$ derived from semiclassical and nonclassical ${}_2F_1$ transformations.

be solution-preserving from the system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, i.e., $\text{gDH}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$, to the system $\dot{x} = Q(x)$, i.e., to $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, is that the respective parameter vectors be restricted and related as specified in Table 4.

Proof. *Mutatis mutandis*, the same as that of Theorem 3.1. In each of the cases $\mathbf{12}_{\mathbf{bq}}, \mathbf{12}_{\mathbf{c}}$ and $\mathbf{10}, \mathbf{24}_{\mathbf{c}}$, the useful formula (3.10) can be used to compute Φ from R , and in particular the rational function $x_1 = \Sigma_1/d$ of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. \square

In fact, the coefficients of each linear polynomial $\hat{\Sigma}_1$ of Table 5 are the multiplicities with which $\tilde{t} = 0, 1, \infty$ are taken by the respective covering map R to $t = 0$, much as with the polynomials Σ_1 of Table 3. But for these semiclassical and nonclassical covering maps, a new feature enters: one or more ordinary points on $\mathbb{P}_{\tilde{t}}^1$ are also mapped to $t = 0$. For instance, for $\mathbf{10}$ one has the factor $32 - 81\tilde{t}$ in P_1 (see Table 2), which is zero at the ordinary point $\tilde{t} = 32/81$. This factor is responsible for Υ equaling $32\tilde{x}_1 + 49\tilde{x}_2 - 81\tilde{x}_3$ in Table 5. The polynomial Υ is of degree 1 for $\mathbf{12}_{\mathbf{bq}}, \mathbf{12}_{\mathbf{c}}$ and $\mathbf{10}$, but of degree 3 for $\mathbf{24}_{\mathbf{c}}$, due to $R^{-1}(0) \subset \mathbb{P}_{\tilde{t}}^1$ comprising three ordinary points, i.e. the roots of $1 + 5\tilde{t} - 8\tilde{t}^2 + \tilde{t}^3$, rather than only one.

Another observation is that up to scaling by $\tilde{c} = c$, in the cases $\mathbf{12}_{\mathbf{bq}}, \mathbf{12}_{\mathbf{c}}$ and $\mathbf{10}, \mathbf{24}_{\mathbf{c}}$ the parameter vectors of the two gDH systems $\dot{x} = Q(x)$, $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ are completely specified. (See Table 4.) In each case $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{c}/2$ and $b_1 = b_2 = b_3 = c/2$, so the solution-preserving maps of Theorem 3.2 are maps between DH systems, *only*. The correspondences $(\alpha_1, \alpha_2, \alpha_3) \leftarrow (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ between the angular parameters of the DH systems, if proper, have already appeared in Tables 1 and 2. For instance, $\mathbf{24}_{\mathbf{c}}$ yields a rational solution-preserving map from $\text{DH}(\frac{1}{7}, \frac{1}{7}, \frac{1}{7})$ to $\text{DH}(\frac{1}{7}, \frac{1}{3}, \frac{1}{2})$.

The relationships among the solution-preserving maps of Theorems 3.1 and 3.2 are displayed in Figure 1, a composition graph. It reveals how a single gDH system (the ‘root’ at the bottom, parametrized by a, b, c) can give rise to many others, by being lifted along the classical covering maps. Each node is labeled by the parameter vector $(\alpha_1, \alpha_2, \alpha_3 | c)$ of a proper DH system, and in all but two cases by the parameter vector $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ of a gDH system that specializes to the proper DH system if the root is constrained by $b = c/2 \neq 0$ and $c - 3a \neq 0$, i.e., if the root is specialized to $\text{DH}(\alpha, \frac{1}{3}, \frac{1}{2} | c)$. Each directed edge of the graph is a solution-preserving map based on $\mathbf{2}, \mathbf{3}, \mathbf{4}$, or $\mathbf{3}_{\mathbf{c}}$. The map may include Möbius transformations that permute singular points, and hence solution components.

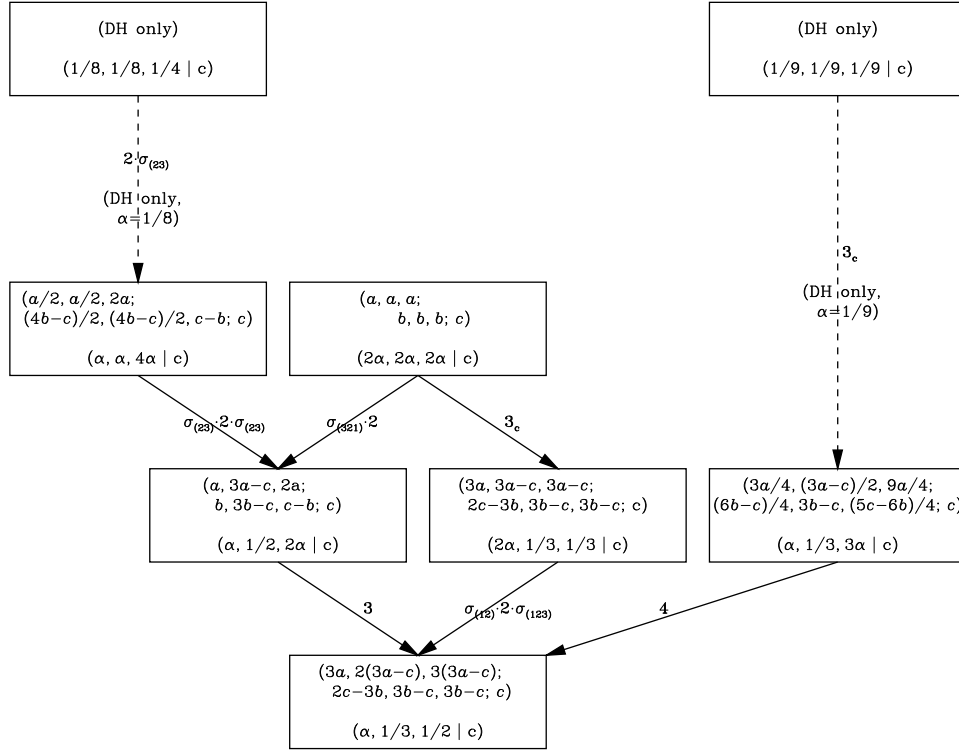


FIGURE 1. A directed graph of gDH systems with free parameters (a, b, c) , which are related by solution-preserving maps and specialize to proper DH systems with free parameters α and c .

For each node in the graph, the DH parameters $(\alpha_1, \alpha_2, \alpha_3)$ were taken from Table 1, and the gDH parameter vector $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ if any was computed from Table 4. For instance, the node atop the ‘diamond’ is the 6_c node, for which $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ is $(a, a, a; b, b, b; c)$. When $b = c/2 \neq 0$ and $c - 3a \neq 0$, the gDH system with this parameter vector specializes to $\text{DH}(2\alpha, 2\alpha, 2\alpha | c)$, where (cf. (1.8)) the angular parameter α is defined by $2\alpha = -a/(c - 3a)$ and must satisfy $\alpha \neq \frac{1}{6}$ if the DH system is to be proper. The left and right sides of the diamond come from $6_c \sim 3 \circ 2 \sim 2 \circ 3_c$, which with Möbius transformations included is

$$(3.33) \quad 6_c = 3 \circ \sigma_{(321)} \circ 2 = \sigma_{(12)} \circ 2 \circ \sigma_{(123)} \circ 3_c.$$

Here $\sigma_{(321)}$, $\sigma_{(12)}$, $\sigma_{(123)}$ permute the singular points $0, 1, \infty$, and hence the gDH components.

The upper left, resp. right nodes are the $12_{bq}, 12_c$ nodes, and the $12_{bq}, 12_c$ maps are functional compositions leading down from them to the root. Those two nodes are ‘DH only,’ as indicated, and moreover are present only if $\alpha = \frac{1}{8}$, resp. $\alpha = \frac{1}{9}$. The decompositions $12_{bq} \sim 3 \circ 2 \circ 2$ and $12_c \sim 4 \circ 3_c$ can be seen.

Nodes $(\alpha_1, \alpha_2, \alpha_3 | c)$ coming from the nonclassical coverings $10, 24_c$ could be added, but are omitted due to lack of space. The $10, 24_c$ nodes are DH-only and

require $\alpha = \frac{1}{7}$. Their angular parameters $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{7}, \frac{1}{3}, \frac{2}{7})$ and $(\frac{1}{7}, \frac{1}{7}, \frac{1}{7})$ were given in Table 2. From each, a directed edge extends to the root node $(\alpha, \frac{1}{3}, \frac{1}{2} | c) = (\frac{1}{7}, \frac{1}{3}, \frac{1}{2} | c)$.

It was mentioned in §3.2 that there are five nonclassical coverings that are not defined over \mathbb{Q} , and will not be considered in any detail. They can be denoted by **6'**, **8**, **9**, **10'**, **18**, after their degrees. The final four of these add DH-only nodes to Figure 1, which are present only if $\alpha = \frac{1}{7}, \frac{1}{7}, \frac{1}{8}, \frac{1}{7}$, respectively. The angular parameters of these nodes are $(\frac{1}{7}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{7}, \frac{1}{2}, \frac{1}{7})$, $(\frac{1}{8}, \frac{1}{3}, \frac{1}{8})$, $(\frac{1}{7}, \frac{1}{7}, \frac{2}{7})$. From each, an edge extends to the root node $(\alpha, \frac{1}{3}, \frac{1}{2} | c)$. The **6'** map is anomalous: it maps $\text{DH}(\frac{1}{4}, \frac{1}{5}, \frac{1}{4} | c)$ to $\text{DH}(\frac{1}{4}, \frac{1}{5}, \frac{1}{2} | c)$, and does not fit into the framework of the figure.

3.4. Maps between non-gDH HQDS's. The focus thus far in §3 has been on gDH systems, and on the consequences of the hypergeometric integration scheme implicit in Theorem 2.1 for the existence of rational solution-preserving maps Φ between them. But the same maps Φ , applied to certain HQDS's that are not of the gDH form, can yield images that are at least HQDS's.

Suppose that $x = \Phi(\tilde{x})$ is solution-preserving from a gDH system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ to a HQDS $\dot{x} = Q(x)$. Suppose also that a HQDS $\dot{\tilde{x}}' = \tilde{Q}'(\tilde{x}')$, not necessarily a gDH system, is linearly equivalent to $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$, i.e., that there is some $\tilde{T} \in GL(3, \mathbb{C})$ such that $\tilde{x}' = \tilde{T}\tilde{x}$ and $\tilde{x} = \tilde{T}^{-1}\tilde{x}'$, by which is meant that the quadratic vector fields \tilde{Q}', \tilde{Q} satisfy $\tilde{Q}' = \tilde{T} \circ \tilde{Q} \circ \tilde{T}^{-1}$. The corresponding non-associative algebras $\tilde{\mathfrak{A}}', \tilde{\mathfrak{A}}$ will then be isomorphic. Clearly $x' = (\Phi \circ T^{-1})(\tilde{x}')$ will be solution-preserving from $\dot{\tilde{x}}' = \tilde{Q}'(\tilde{x}')$ to $\dot{x} = Q(x)$. Moreover if Φ satisfies $\Phi = T \circ \Phi \circ \tilde{T}^{-1}$ for some $T, \tilde{T} \in GL(3, \mathbb{C})$, then Φ itself will map the HQDS $\dot{\tilde{x}}' = \tilde{Q}'(\tilde{x}')$ to a HQDS $\dot{x}' = Q'(x')$, the vector field Q' being defined by $Q' = T \circ Q \circ T^{-1}$.

The following two theorems give examples of non-gDH HQDS's on which the maps Φ coming from the coverings **2**, **3_c** and **6_c** are solution-preserving to other HQDS's. In some cases this is because they are linearly equivalent to the gDH systems of Theorem 3.1 by $\tilde{x}' = \tilde{T}\tilde{x}$ for some \tilde{T} , and Φ satisfies $\Phi = T \circ \Phi \circ \tilde{T}^{-1}$ for some T . The map $x = \Phi(\tilde{x})$ coming from **2** was given in (3.1a), and the 'cyclic' ones coming from **3_c**, **6_c** can be read off from Table 3. Explicitly, the **3_c** map is

$$(3.34) \quad x_1 = \frac{1}{3}S_1, \quad x_2 = \frac{1}{3}\left(S_1 - \frac{z^2}{\bar{z}}\right), \quad x_3 = \frac{1}{3}\left(S_1 - \frac{\bar{z}^2}{z}\right),$$

and the **6_c** map is

$$(3.35) \quad x_1 = \frac{1}{3}S_1, \quad x_2 = \frac{1}{3}\left(S_1 - \frac{z^3 + \bar{z}^3}{2z\bar{z}}\right), \quad x_3 = \frac{1}{3}\left(S_1 - \frac{2(z\bar{z})^2}{z^3 + \bar{z}^3}\right),$$

$$= \frac{9S_3 - S_1S_2}{2(3S_2 - S_1^2)}, \quad = \frac{9S_1S_3 - 6S_2^2 + S_1^2S_2}{27S_3 - 9S_1S_2 + 2S_1^3}.$$

Here

$$(3.36) \quad S_1 := \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3, \quad S_2 := \tilde{x}_1\tilde{x}_2 + \tilde{x}_2\tilde{x}_3 + \tilde{x}_3\tilde{x}_1, \quad S_3 := \tilde{x}_1\tilde{x}_2\tilde{x}_3$$

are the elementary symmetric polynomials in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, and

$$(3.37) \quad z := \tilde{x}_1 + \omega\tilde{x}_2 + \bar{\omega}\tilde{x}_3, \quad \bar{z} := \tilde{x}_1 + \bar{\omega}\tilde{x}_2 + \omega\tilde{x}_3$$

are the cyclic relative invariants, ω being a primitive cube root of unity. By $\mathfrak{Z}_2, \mathfrak{Z}_3, \mathfrak{S}_3 < GL(3, \mathbb{C})$ will be meant the subgroups generated by $\tilde{x}_1 \leftrightarrow \tilde{x}_3$ and $\tilde{x}_1 \rightarrow \tilde{x}_2 \rightarrow \tilde{x}_3 \rightarrow \tilde{x}_1$, and the symmetric group on $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$.

A general HQDS $(\mathbb{C}^3, \dot{\tilde{x}} = \tilde{Q}(\tilde{x}))$, with 18 parameters, can be written as

$$(3.38) \quad \begin{cases} \dot{\tilde{x}}_1 = a_{11}\tilde{x}_1^2 + a_{12}\tilde{x}_2^2 + a_{13}\tilde{x}_3^2 + b_{11}\tilde{x}_2\tilde{x}_3 + b_{12}\tilde{x}_3\tilde{x}_1 + b_{13}\tilde{x}_1\tilde{x}_2, \\ \dot{\tilde{x}}_2 = a_{21}\tilde{x}_1^2 + a_{22}\tilde{x}_2^2 + a_{23}\tilde{x}_3^2 + b_{21}\tilde{x}_2\tilde{x}_3 + b_{22}\tilde{x}_3\tilde{x}_1 + b_{23}\tilde{x}_1\tilde{x}_2, \\ \dot{\tilde{x}}_3 = a_{31}\tilde{x}_1^2 + a_{32}\tilde{x}_2^2 + a_{33}\tilde{x}_3^2 + b_{31}\tilde{x}_2\tilde{x}_3 + b_{32}\tilde{x}_3\tilde{x}_1 + b_{33}\tilde{x}_1\tilde{x}_2. \end{cases}$$

The special case $a_{ij} = a_i\delta_{ij}$, $b_{ij} = (2a_i - c)\delta_{ij} - a_i + b_j$ is the gDH case, as was mentioned in the Introduction; cf. (1.4).

Theorem 3.3. *Let $(\mathbb{C}^3, \dot{\tilde{x}} = \tilde{Q}(\tilde{x}))$ be a HQDS of the form (3.38) that is invariant under \mathfrak{Z}_2 , resp. \mathfrak{Z}_3 , resp. \mathfrak{S}_3 , i.e., satisfying $a_{\pi(i),\pi(j)} = a_{ij}$ and $b_{\pi(i),\pi(j)} = b_{ij}$ for each supported permutation π of 1, 2, 3. In the \mathfrak{Z}_2 case it is also required that*

$$\begin{aligned} & [(a_{11} + a_{12} + a_{13}) - 2(a_{21} + a_{22} + a_{23}) + (a_{31} + a_{32} + a_{33})] \\ & + [(b_{11} + b_{12} + b_{13}) - 2(b_{21} + b_{22} + b_{23}) + (b_{31} + b_{32} + b_{33})] = 0. \end{aligned}$$

Then, the $\mathbf{2}$ map (3.1a), resp. the $\mathbf{3_c}$ map (3.34), resp. the $\mathbf{6_c}$ map (3.35), will take $(\mathbb{C}^3, \dot{\tilde{x}} = \tilde{Q}(\tilde{x}))$ to some HQDS $(\mathbb{C}^3, \dot{x} = Q(x))$.

These claims are readily confirmed with the aid of a computer algebra system. The image HQDS's $\dot{x} = Q(x)$ in the $\mathbf{3_c}$ and $\mathbf{6_c}$ cases are made explicit in the following; the $\mathbf{2}$ case is left to the reader.

Theorem 3.4. *Let a cyclically symmetric (i.e., \mathfrak{Z}_3 -invariant) HQDS be defined by*

$$(3.39) \quad \begin{aligned} \dot{\tilde{x}}_i &= -\tilde{a}(\tilde{x}_i - \tilde{x}_j)(\tilde{x}_k - \tilde{x}_i) + \tilde{b}(\tilde{x}_2\tilde{x}_3 + \tilde{x}_3\tilde{x}_1 + \tilde{x}_1\tilde{x}_2) - \tilde{c}\tilde{x}_j\tilde{x}_k \\ &+ \tilde{d}(\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)^2 + \tilde{e}\tilde{x}_i(\tilde{x}_j - \tilde{x}_k) + \tilde{f}(\tilde{x}_j^2 - \tilde{x}_k^2), \end{aligned}$$

for i, j, k equal to each cyclic permutation of 1, 2, 3. The $\mathbf{3_c}$ map (3.34) will take this HQDS to the image HQDS

$$\begin{cases} \dot{x}_1 = -3\tilde{a}(x_1 - x_2)(x_3 - x_1) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_2x_3 + 9\tilde{d}x_1^2, \\ \dot{x}_2 = -(3\tilde{a} - \tilde{c})(x_2 - x_3)(x_1 - x_2) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_3x_1 + 9\tilde{d}x_1^2 \\ \quad + (\omega - \bar{\omega})(x_1 - x_2)[\tilde{e}(x_2 + 2x_3) + \tilde{f}(9x_1 - x_2 - 2x_3)], \\ \dot{x}_3 = -(3\tilde{a} - \tilde{c})(x_3 - x_1)(x_2 - x_3) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_1x_2 + 9\tilde{d}x_1^2 \\ \quad + (\omega - \bar{\omega})(x_3 - x_1)[\tilde{e}(2x_2 + x_3) + \tilde{f}(9x_1 - 2x_2 - x_3)]. \end{cases}$$

If $\tilde{e} = \tilde{f} = 0$ (expanding the invariance group from \mathfrak{Z}_3 to \mathfrak{S}_3), the $\mathbf{6_c}$ map (3.35) can also be applied, yielding the image HQDS

$$\begin{cases} \dot{x}_1 = -3\tilde{a}(x_1 - x_2)(x_3 - x_1) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_2x_3 + 9\tilde{d}x_1^2, \\ \dot{x}_2 = -2(3\tilde{a} - \tilde{c})(x_2 - x_3)(x_1 - x_2) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_3x_1 + 9\tilde{d}x_1^2, \\ \dot{x}_3 = -3(3\tilde{a} - \tilde{c})(x_3 - x_1)(x_2 - x_3) + \mathbf{B}(x_1, x_2, x_3) - \tilde{c}x_1x_2 + 9\tilde{d}x_1^2. \end{cases}$$

In both image HQDS's,

$$\mathbf{B}(x_1, x_2, x_3) := (2\tilde{c} - 3\tilde{b})x_2x_3 + (3\tilde{b} - \tilde{c})x_3x_1 + (3\tilde{b} - \tilde{c})x_1x_2.$$

Remark. Setting $\tilde{d} = \tilde{e} = \tilde{f} = 0$ reduces the initial HQDS to a gDH system with $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}$ and $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}$; cf. (1.4). Hence the terms proportional to $\tilde{a}, \tilde{b}, \tilde{c}$ in each image HQDS can be verified by examining Table 4. It is only the terms proportional to the gDH deformation parameters $\tilde{d}, \tilde{e}, \tilde{f}$ that are new.

Remark. Setting $\tilde{e} = \tilde{f} = 0$ removes the terms from the image under $\mathbf{3}_c$ that break invariance under $x_2 \leftrightarrow x_3$, thereby allowing $\mathbf{3}_c$ to be composed with a further solution-preserving map of quadratic type. But for the map $\mathbf{6}_c$ to be obtained thus, by composing $\mathbf{3}_c$ with $\mathbf{2}$ (which is based on symmetry under $x_1 \leftrightarrow x_3$, not $x_2 \leftrightarrow x_3$), two permutations of components are needed; cf. (3.33).

It follows from the discussion in §3.2 that in each of the ‘nice’ (i.e. Galois) cases $\mathbf{2}, \mathbf{3}_c, \mathbf{6}_c$, there is a simple interpretation of the polynomials $\Sigma_1, \Sigma_2, \Sigma_3$ that appear, according to Theorem 3.1, in the solution-preserving map $x = \Phi(\tilde{x})$. Associated to the respective Galois group $G = \mathfrak{Z}_2, \mathfrak{Z}_3, \mathfrak{S}_3$ there is a ring $I(G)$ of polynomial invariants in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; and $\Sigma_1, \Sigma_2, \Sigma_3$ generate $I(G)$. (The case $G = \mathfrak{Z}_2$ is a bit special: a fourth invariant $\Sigma'_1 := x_2$ must be added.) When $G = \mathfrak{S}_3$, say, this is equivalent to saying that the ring of symmetric polynomials in $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ is generated by the elementary symmetric polynomials S_1, S_2, S_3 . Any HQDS that is invariant under \mathfrak{S}_3 can be taken to a polynomial differential system satisfied by the invariants S_1, S_2, S_3 , via a solution-preserving map.

The significance of Theorems 3.3 and 3.4 is that they reveal that the differential system satisfied by x_1, x_2, x_3 (each a suitably chosen rational function of S_1, S_2, S_3 , or more generally of $\Sigma_1, \Sigma_2, \Sigma_3$) will be a HQDS, like the original system for $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. In the area of 3-dimensional differential systems, this may be a new discovery. In Ref. [42], which introduced and exploited the ring $I(G)$, several HQDS’s were solved, in each case by mapping the HQDS onto a differential system satisfied by a vector of cyclic invariants. But since the invariants were S_1, S_2, S_3 rather than the above x_1, x_2, x_3 , the system was not a HQDS.

Cyclically symmetric HQDS’s have arisen many times in dynamical systems theory. If $\tilde{a} = \tilde{d} = \tilde{f} = 0$, the HQDS (3.39) of Theorem 3.4 becomes the Leonard–May model of cyclic competition among three species [51, 60].³ If moreover $\tilde{b} = \tilde{c} = 0$, so that \tilde{e} is its only nonzero parameter, the HQDS (3.39) becomes the periodic 3-particle Volterra model $\dot{\tilde{x}}_i = \tilde{e} \tilde{x}_i(\tilde{x}_j - \tilde{x}_k)$, or equivalently the 3-particle KM (Kac–van Moerbeke) system [23], which is integrable by elliptic functions [11, §11]. Such models have been studied exhaustively, but it may not have been noticed before that in the case of three particles (or species), the cyclic symmetry group \mathfrak{Z}_3 can be quotiented out in such a way as to yield another HQDS.

Another noteworthy special case of the HQDS (3.39) is the Kasner system $\dot{\tilde{x}}_i = \tilde{x}_j \tilde{x}_k - \tilde{x}_i^2$, which is invariant under \mathfrak{S}_3 . (See [40, 42] and [66, §5.3].) Of course the Kasner system is an (improper) gDH system, with $(a_1, a_2, a_3; b_1, b_2, b_3; c) = -(1, 1, 1; 1, 1, 1; 3)$. But the $\mathbf{3}_c$ half of Theorem 3.4 gives many examples of solution-preserving maps from HQDS’s that are not gDH systems, or linearly equivalent to them. (For instance, the periodic 3-particle Volterra model is not.) The $\mathbf{6}_c$ half is a bit less interesting, since when $\tilde{e} = \tilde{f} = 0$, the HQDS (3.39) turns out always to be gDH-equivalent. But the $\mathbf{6}_c$ half can be generalized in the following way.

Theorem 3.5. *For each classical PE-lifting covering map $t = R(\tilde{t})$ (of degree d) listed in Table 1, i.e., for each of $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{6}_c$ and $\mathbf{3}_c, \mathbf{4}_{\mathbf{bq}}$, the associated rational map $x = \Phi(\tilde{x})$ given in Theorem 3.1 satisfies $\Phi = T_\epsilon \circ \Phi \circ \tilde{T}_\epsilon^{-1}$, where $T_\epsilon, \tilde{T}_\epsilon \in$*

³In the model as usually defined, each $\dot{\tilde{x}}_i$ also includes a term $r\tilde{x}_i$, r being the common species growth rate. The substitutions $\tilde{X}_i = e^{-rt}\tilde{x}_i$, $dT = e^{rt}dt$ remove these linear terms.

$GL(3, \mathbb{C})$ are defined by

$$\begin{aligned} (\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3) &= \tilde{T}_\epsilon(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) := (1 - \epsilon)(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) + \epsilon [\Sigma_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)/d] (1, 1, 1), \\ (x'_1, x'_2, x'_3) &= T_\epsilon(x_1, x_2, x_3) := (1 - \epsilon)(x_1, x_2, x_3) + \epsilon x_1(1, 1, 1), \end{aligned}$$

$\epsilon \neq 1$ being arbitrary. That is, if Φ maps a gDH system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ to a gDH system $\dot{x} = Q(x)$, it also maps a one-parameter deformation $\dot{\tilde{x}}' = \tilde{Q}'(\tilde{x}')$ of the former to a one-parameter deformation $\dot{x}' = Q'(x')$ of the latter, the deformed (linearly equivalent) vector fields being $\tilde{Q}' := \tilde{T}_\epsilon \circ \tilde{Q} \circ \tilde{T}_\epsilon^{-1}$ and $Q' := T_\epsilon \circ Q \circ T_\epsilon^{-1}$.

Proof. As in the proof of Theorem 3.1, let $y_k := x_i - x_j$, etc. What is to be shown is that if $\tilde{x}' = \tilde{T}\tilde{x}$ and $x' = Tx$, then $x' = \Phi(\tilde{x}')$. But $\tilde{y}'_k = (1 - \epsilon)\tilde{y}_k$ and $y'_k = (1 - \epsilon)y_k$. By the formula (3.11), each y_k is a homogeneous degree-1 rational function of $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$; thus each y'_k can be represented as the *same* function of $\tilde{y}'_1, \tilde{y}'_2, \tilde{y}'_3$. Hence it suffices to prove, say, that x'_1 is the same function of $\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3$ as x_1 is of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; i.e. (see the statement of Theorem 3.1), that given $x_1 = \Sigma_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, one also has $x'_1 = \Sigma_1(\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3)$. But by definition, $x'_1 = x_1$; and for each covering map, Σ_1 is linear in its arguments with coefficients that sum to d (since they are multiplicities, as was remarked after Theorem 3.1). It follows that $x'_1 = \Sigma_1(\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3)$. \square

Some elementary further calculations reveal that for each choice of covering map, each of the deformed quadratic vector fields \tilde{Q}', Q' of Theorem 3.5 can be written as the vector field of a gDH system, of the familiar type shown in (1.4), plus a deformation field of a simple form. This is summarized in the following.

Theorem 3.6. *If a rational map $x = \Phi(\tilde{x})$ associated by Theorem 3.1 to one of the classical covering maps **2, 3, 4, 6, 6_c** and **3_c, 4_{bq}** (of degree d) is solution-preserving from $\text{gDH}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{c})$ to $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, then the same will be true if the two gDH systems are deformed: a term $\delta [\Sigma_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)/d]^2$ being added to the expression for each $\dot{\tilde{x}}_i$, and a term δx_i^2 to that for each \dot{x}_i . The deformation parameter $\delta \in \mathbb{C}$ is arbitrary.*

It should be stressed that the deformed systems of this theorem are linearly equivalent to gDH ones. The **6_c** case, in which $d = 6$ and $\Sigma_1 = 2\tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3$ by Table 3, subsumes the **6_c** half of Theorem 3.4. (In the statement of that theorem, the deformation parameter δ appeared as $9\tilde{d}$.)

3.5. Equivalences among gDH systems. Distinct gDH systems $\dot{x} = Q(x)$ and $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ of the form (1.4) may be linearly equivalent, i.e., may be related by $\tilde{x} = T x$ for some $T \in GL(3, \mathbb{C})$. That is, the vector fields Q, \tilde{Q} may satisfy $\tilde{Q} = T \circ Q \circ T^{-1}$. One case is when T permutes the components (x_1, x_2, x_3) into $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, perhaps with scaling; but there are also nontrivial equivalences, summarized in Theorem 3.7 below. The existence of nontrivial equivalences has implications for the (as yet incomplete) classification up to isomorphism of 3-dimensional commutative, non-associative algebras. Even if one restricts oneself to algebras \mathfrak{A} of gDH type, one must quotient out a group action on the space of gDH parameters $(a_1, a_2, a_3; b_1, b_2, b_3; c)$. A fundamental domain for this action will need to be computed, as in the classification of 2-dimensional non-associative algebras [50].

Nontrivial linear equivalences between gDH systems are an algebraic matter, but interestingly, explicit formulas for such equivalences follow directly from the PE-based integration scheme of § 2.2. The scheme employed offset exponents $(\nu_1, \nu'_1;$

$\nu_2, \nu'_2; \nu_3, \nu'_3$), rationally related to $(a_1, a_2, a_3; b_1, b_2, b_3)$ and c by formulas given in Theorem 2.1; and another parameter, \bar{n} . In fact the 7-dimensional linear space of gDH systems can alternatively be parametrized by $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}; c)$. By Fuchs's relation, the six offset exponents are not independent; see (3.19).

There is a natural group \mathfrak{G} of gDH equivalences, which is generated by (1) permutations of the 3-set $\{(\nu_1, \nu'_1), (\nu_2, \nu'_2), (\nu_3, \nu'_3)\}$, and (2) the transpositions $\nu_i \leftrightarrow \nu'_i$, $i = 1, 2, 3$. The group \mathfrak{G} is clearly isomorphic to the group of *signed* permutations of a 3-set, which is the order-48 Coxeter group, \mathcal{B}_3 . Any element of \mathcal{B}_3 can be written in a sign-annotated version of the disjoint cycle representation used for elements of \mathfrak{S}_3 . For instance, $[1_+ 2_-][3_-]$ signifies $1 \mapsto 2$, $2 \mapsto -1$, $3 \mapsto -3$. (Positively signed 1-cycles are typically omitted.) The group \mathcal{B}_3 is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_3$, i.e. to a semidirect product $(\mathbb{Z}_2)^3 \rtimes \mathfrak{S}_3$, where the normal subgroup $(\mathbb{Z}_2)^3$ is generated by the involutions $\nu_i \leftrightarrow \nu'_i$, $i = 1, 2, 3$. In fact \mathcal{B}_3 turns out to be isomorphic to $\mathbb{Z}_2 \times \mathfrak{S}_4$, where the \mathbb{Z}_2 factor is generated by the involution $[1_-][2_-][3_-]$, i.e., by $\nu_i \leftrightarrow \nu'_i$ ($\forall i$); though this will not be needed.

Theorem 3.7. *There is an order-48 gDH-stabilizing group $\mathfrak{G} \cong \mathcal{B}_3 \cong \mathbb{Z}_2 \wr \mathfrak{S}_3$, which acts on the gDH parameter space as follows. Positive-signed elements of \mathfrak{G} , i.e., pure permutations, act by permuting the pairs (a_1, b_1) , (a_2, b_2) , (a_3, b_3) ; and the associated linear equivalence $\bar{x} = Tx$ permutes the components x_1, x_2, x_3 into $\bar{x}_1, \bar{x}_2, \bar{x}_3$. Also, for $i = 1, 2, 3$, the involution $[i_-] \in \mathfrak{G}$ performs the map $(a_1, a_2, a_3; b_1, b_2, b_3; c) \mapsto (\bar{a}_1, \bar{a}_2, \bar{a}_3; \bar{b}_1, \bar{b}_2, \bar{b}_3; \bar{c})$ given by*

$$(3.40) \quad \begin{cases} \bar{a}_i = c - (c - a_i) \left(\frac{2c - b_1 - b_2 - b_3}{c - a_i - b_i} \right), \\ \bar{a}_j = a_j \left(\frac{2c - b_1 - b_2 - b_3}{c - a_i - b_i} \right); \\ \bar{b}_i = -(c + a_i - b_1 - b_2 - b_3) \left(\frac{2c - b_1 - b_2 - b_3}{c - a_i - b_i} \right), \\ \bar{b}_j = -(c - b_1 - b_2 - b_3) + (c - b_i - b_k) \left(\frac{2c - b_1 - b_2 - b_3}{c - a_i - b_i} \right) \end{cases}$$

with $\bar{c} = c$, the associated linear equivalence $\bar{x} = T_{[i_-]}x$ being

$$(3.41) \quad \begin{cases} \bar{x}_i = x_i, \\ \bar{x}_j = x_i + \left(\frac{c - a_i - b_i}{2c - b_1 - b_2 - b_3} \right) (x_j - x_i). \end{cases}$$

In the preceding, j stands for each element of $\{1, 2, 3\}$ other than i , and k for the third element.

Proof. That the substitution $\bar{x} = T_{[i_-]}x$ of (3.41) maps $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$ to $\text{gDH}(\bar{a}_1, \bar{a}_2, \bar{a}_3; \bar{b}_1, \bar{b}_2, \bar{b}_3; \bar{c})$ can be checked by hand. For a more comprehensible derivation of (3.40) and (3.41) based on Theorem 2.1, reason as follows.

From $(a_1, a_2, a_3; b_1, b_2, b_3)$ and c , compute $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$ by Eqs. (2.16) of that theorem. Perform the transposition $\nu_i \leftrightarrow \nu'_i$, and by applying Eqs. (2.14), reverse the direction of computation to obtain $(\bar{a}_1, \bar{a}_2, \bar{a}_3; \bar{b}_1, \bar{b}_2, \bar{b}_3)$. After algebraic simplification of rational functions, one finds (3.40).

To derive (3.41), use the expressions given in Theorem 2.1 for x_1, x_2, x_3 as logarithmic derivatives of PE solutions. With a common constant of proportionality,

these are (see (2.17))

$$(3.42) \quad x_i \propto (d/d\tau) \log(\Delta_i f) = (d/d\tau) \log [(t - t_i)^{\mu_j + \mu_k} (t - t_j)^{-\mu_j} (t - t_k)^{-\mu_k} f],$$

for i, j, k any cyclic permutation of $1, 2, 3$. Now, let i once again mean the index at which the interchange $\nu_i \leftrightarrow \nu'_i$ of offset exponents (i.e., the interchange $\mu_i \leftrightarrow \mu'_i$ of exponents) takes place, producing \bar{x} from x . It then follows from (3.42) that

$$(3.43) \quad \begin{cases} \bar{x}_i = x_i, \\ \bar{x}_j = x_i + \left(\frac{\mu'_i + \mu_j + \mu_k}{\mu_i + \mu_j + \mu_k} \right) (x_j - x_i), \end{cases}$$

where j now stands for each element of $\{1, 2, 3\}$ other than i , and k for the third element. Using $\mu'_i = \mu_i + \alpha_i$ and $\mu_l = \nu_l - (2\kappa_l - 1)\bar{n}$, $l = 1, 2, 3$, one has

$$(3.44) \quad \frac{\mu'_i + \mu_j + \mu_k}{\mu_i + \mu_j + \mu_k} = \frac{\alpha_i + \nu_1 + \nu_2 + \nu_3 + \bar{n}}{\nu_1 + \nu_2 + \nu_3 + \bar{n}},$$

as $\kappa_1 + \kappa_2 + \kappa_3 = 1$. Expressing \bar{n} and ν_1, ν_2, ν_3 and α_i in terms of $(a_1, a_2, a_3; b_1, b_2, b_3; c)$, by Eqs. (2.16), yields (3.41) from (3.43). \square

In its action on the gDH parameter space, \mathfrak{G} is an order-48 *algebraic group*: a collection of rational maps that act on points $(a_1, a_2, a_3; b_1, b_2, b_3; c)$. Hence any gDH system lies on an orbit of up to 48 gDH systems. Some orbits are of size strictly less than 48. For instance, one may have for some $i \neq j$ that $a_i = a_j$ and $b_i = b_j$, so that $(\nu_i, \nu'_i) = (\nu_j, \nu'_j)$, in which case $[i+j_+] \in \mathfrak{G}$, which performs the interchange $(\nu_i, \nu'_i) \leftrightarrow (\nu_j, \nu'_j)$, will leave the gDH system invariant. Also, for some $g \in \mathfrak{G}$ the map $(a_1, a_2, a_3; b_1, b_2, b_3; c) \rightarrow (\bar{a}_1, \bar{a}_2, \bar{a}_3; \bar{b}_1, \bar{b}_2, \bar{b}_3; \bar{c})$ may involve a division by zero. That is, the transformed gDH system may be undefined.

One should also note that in exceptional cases, applying the element $[i_-] \in \mathfrak{G}$, i.e., the transformation $\nu_i \leftrightarrow \nu'_i$ or equivalently the negation $\alpha_i \mapsto -\alpha_i$, may convert a proper gDH system to an improper one, or vice versa. (Recall Definition 2.2.)

To understand \mathfrak{G} better, it is useful to consider its action on DH systems, for which $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ equals $(a_1, a_2, a_3; b, b, b; c)$ with $b = c/2$. (Proper) DH systems of the form (1.7) are parametrized by $(\alpha_1, \alpha_2, \alpha_3 | c)$, where the angular parameters satisfy the condition that $\rho^{-1} := (1 - \alpha_1 - \alpha_2 - \alpha_3)/2$ is nonzero (and also, $c \neq 0$). The group \mathfrak{G} acts on the DH parameter space as a group of signed permutations: the orbit of $(\alpha_1, \alpha_2, \alpha_3 | c)$ consists of the 48 points $(\pm\alpha_{1'}, \pm\alpha_{2'}, \pm\alpha_{3'} | c)$, where $\alpha_{1'}, \alpha_{2'}, \alpha_{3'}$ is a permutation of $\alpha_1, \alpha_2, \alpha_3$. This is because the transposition $\nu_i \leftrightarrow \nu'_i$ specializes to the negation $\alpha_i \mapsto -\alpha_i$. Owing to (unsigned) angles appearing multiple times, a \mathfrak{G} -orbit may contain fewer than 48 distinct DH systems. In exceptional cases a proper DH system can be transformed to an improper one.

The following explicit formulas dealing with the involution $[1_-][2_-][3_-] \in \mathfrak{G}$, which performs $\nu_i \leftrightarrow \nu'_i$ ($\forall i$), will be used in the construction of gDH representations of the solutions of Chazy equations.

Proposition 3.8. *The linear equivalence $\bar{x} = T_{[1_-][2_-][3_-]}x$, acting on any solution $x = x(\tau)$ of $\text{gDH}(a, a, a; b, b, b; c)$, is the circulant map*

$$\bar{x}_i = x_i - \frac{c + a - 2b}{2c - 3b} (2x_i - x_j - x_k), \quad i = 1, 2, 3,$$

in which j, k are the elements of $\{1, 2, 3\}$ other than i . The corresponding transformation of gDH systems is $\text{gDH}(a, a, a; b, b, b; c) \mapsto \text{gDH}(\bar{a}, \bar{a}, \bar{a}; b, b, b; c)$, where

$$\bar{a} = a + \frac{(c - 3a)(c + a - 2b)}{c + 3a - 3b}.$$

It is assumed that the transformed gDH system is not undefined, i.e., that there is no division by zero.

Proof. By a lengthy direct computation, i.e. by successively applying the formulas for $[i_-]$, $i = 1, 2, 3$, given in Theorem 3.7. \square

The group \mathfrak{G} can be used to create new solution-preserving maps between non-isomorphic gDH systems. If $g_1, g_2 \in \mathfrak{G}$ and $x = \Phi(\tilde{x})$ is one of the rational solution-preserving maps of Theorem 3.1, from some gDH system $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ to another gDH system $\dot{x} = Q(x)$, then $\bar{\Phi} := g_2 \circ \Phi \circ g_1^{-1}$ will be solution-preserving from $\dot{\tilde{x}} = \tilde{Q}(\tilde{x})$ to $\dot{\bar{x}} = \bar{Q}(\bar{x})$, where $\tilde{\bar{x}}, \bar{x}$ come from \tilde{x}, x by $\tilde{\bar{x}} = T_{g_1}\tilde{x}$ and $\bar{x} = T_{g_2}x$. In this way, up to $48^2 = 2304$ rational maps $\bar{\Phi}$ can be produced. Of course maps obtained by mere permutation of components (of $\tilde{\bar{x}}$ and/or \bar{x}) are of relatively little interest. However, each map Φ of Theorem 3.1 lies on an orbit of up to $8^2 = 64$ significantly different maps $\bar{\Phi}$, obtained by acting on \tilde{x} and/or x by elements of \mathfrak{G} that are products of zero or more negatively signed 1-cycles, such as the symmetric involution $[1_-][2_-][3_-]$ (the product of all three).

If one focuses on the hypergeometric integration scheme of § 3.2, one can interpret $\mathfrak{G} \cong \mathcal{B}_3$ as the automorphism group of the Papperitz equation (2.1b), or of its P-symbol (2.1a). (That group is generated by permutations of the singular points t_1, t_2, t_3 and interchanges $\mu_1 \leftrightarrow \mu'_1, \mu_2 \leftrightarrow \mu'_2, \mu_3 \leftrightarrow \mu'_3$ of exponents.) The group \mathfrak{G} is therefore closely related to the automorphism group of the Gauss hypergeometric equation (2.2b), or of its P-symbol (2.2a). As usually defined, the latter group is the order-24 Coxeter group of *even-signed* permutations of a 3-set, which is isomorphic to \mathfrak{S}_4 . It can be viewed as the origin of Kummer's well-known set of 24 local solutions of the GHE, each of which is expressed in terms of ${}_2F_1$, and four of which are in fact identical to ${}_2F_1$ [48].

The many alternative rational solution-preserving maps $\bar{\Phi} := g_2 \circ \Phi \circ g_1^{-1}$ that can be produced from each Φ , i.e. from each of the PE-lifting maps of Table 1, have a hypergeometric interpretation. Like Φ , each is associated to a classical ${}_2F_1$ transformation that is based on a covering $\bar{R}: \mathbb{P}_{\tilde{t}}^1 \rightarrow \mathbb{P}_t^1$, but \bar{R} may differ from the covering R of the table because of pre- and/or post-composition with Möbius transformations that permute the singular points $\tilde{t} = 0, 1, \infty$ and/or $t = 0, 1, \infty$. There may also be interchanges of exponents. For example, there are quadratic solution-preserving maps that are associated to the classical quadratic transformations of ${}_2F_1$ other than the canonical one, (3.6). But, each of the other quadratic transformations can be obtained from (3.6) by P-symbol manipulations [4].

4. GDH SYSTEMS OF PAINLEVÉ TYPE

A gDH system (1.4) may have the Painlevé property (PP), according to which no local solution has a branch point; at least, not one that is movable, with a location depending on the choice of initial condition $x(\tau_0) = x^0$.

The (proper) DH case is fully understood. A proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3)$ has the PP if and only if $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$, where each N_i is a nonzero

integer or ∞ . The most familiar subcase is when each N_i is a *positive* integer or ∞ . If $\alpha_1 + \alpha_2 + \alpha_3 < 1$, any noncoincident solution $x = x(\tau)$ of (1.4) is confined by a natural barrier (a ‘wall of poles’) to a half-plane or disk, as mentioned in the Introduction. This maximal domain of definition is solution-dependent, i.e., depends on the initial condition. If the domain is $\text{Im } \tau > 0$, then $x = x(\tau)$ will be a vector of quasi-modular forms for a triangle group $\Delta(N_1, N_2, N_3) < PSL(2, \mathbb{R})$.

Non-DH gDH systems have no modular interpretation: they are affine-covariant but not projective-covariant. But one can prove the following sophisticated classification theorem (Theorem 4.1, including Table 6) on proper non-DH gDH systems that have the PP. Interestingly, many such systems are related by rational solution-preserving maps $x = \Phi(\tilde{x})$ of the types constructed in §3.

The classification is made possible by the proper gDH \leftrightarrow gSE (generalized Schwarzian equation) correspondence of Theorem 2.9. Recall that this correspondence is based on the maps $t(\cdot) \mapsto x(\cdot)$, $x(\cdot) \mapsto t(\cdot)$ of Eqs. (2.32), (2.34), the latter reducing to $t = -(x_2 - x_3)/(x_1 - x_2)$ if $(t_1, t_2, t_3) = (0, 1, \infty)$. Theorem 4.1 is accordingly a corollary of the classification of the proper gSE’s of the form (2.41) that have the PP but are not SE’s, which was begun by Garnier [24] and completed by Carton-LeBrun [12], using a rigorous version of Painlevé’s α -method. Garnier and Carton-LeBrun chose $(0, 1, \infty)$ as the gSE singular points, with no loss of generality.

Theorem 4.1 is followed, for purposes of illustration, by the explicit integration of several proper non-DH gDH systems with the PP. In fact, each of the systems in Table 6 is integrable; this follows from the integration of all non-SE gSE’s with the PP, also performed by Garnier and Carton-LeBrun. (See Theorem 4.3, including Table 7, and the detailed Examples 4.4–4.7.) In most cases, the noncoincident gDH solutions $x = x(\tau)$ are doubly or at least simply periodic in τ . In Example 4.5, an application to the integration of Chazy-XI is indicated.

Theorem 4.1. *The non-DH gDH systems with the Painlevé property, which are proper in the sense of Definition 2.2, i.e., satisfy (i) $c \neq 0$, (ii) $2c - b_1 - b_2 - b_3 \neq 0$, and (iii) $c - a_1 - a_2 - a_3 \neq 0$, are listed up to linear equivalence in Table 6.*

On each line a vector $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ is given, as are the elements of the alternative (birationally equivalent) vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}; c)$, and the angular parameters $\alpha_i := \nu'_i - \nu_i$. (The freedom in the choice of c is exploited to make each of $a_1, a_2, a_3; b_1, b_2, b_3$ an integer.) In each non-DH ($\bar{n} \neq 1/2$) case with the PP, \bar{n} equals $(n+1)/n$, with n an integer ($n \neq 0, -1, -2$) or ∞ . The table is partitioned according to n .

The list is complete, up to the multiplication of $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ by a nonzero constant, and up to the action on the gDH parameter space of the order-48 isomorphism group \mathfrak{S} of Theorem 3.7, which is generated by permutations of the components $i = 1, 2, 3$, and the three interchanges $\nu_i \leftrightarrow \nu'_i$, $i = 1, 2, 3$. Thus each line may stand for up to 48 distinct $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$; and up to 48 distinct $(a_1, a_2, a_3; b_1, b_2, b_3; c)$, each of which can be multiplied by a nonzero constant.

The classification data in Table 6 were extracted from Tables I–VII of [12], with some labor. The following brief explanation of the results of [12], formal rather than rigorous, may be useful. Consider a gSE of the form (2.41) with parameters $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$, the parameters associated to $t = t_i$ being ν_i, ν'_i , and the Fuchsian condition being $\sum_{i=1}^3 (\nu_i + \nu'_i) = 1 - 2\bar{n}$. Of interest is the behavior of a

TABLE 6. All proper non-DH gDH systems with the Painlevé property, up to isomorphism. ($r \in \mathbb{Z} \setminus \{0\}$ is arbitrary.) Several components x_i satisfying Chazy equations are indicated in brackets.

system	$(a_1, a_2, a_3; b_1, b_2, b_3; c)$	$(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$	$(\alpha_1, \alpha_2, \alpha_3)$	$(r_1, r'_1; r_2, r'_2; r_3, r'_3)$	preimage(s), [notes]
n an arbitrary integer ($\neq 0, -1, -2$), with $\bar{n} = (n+1)/n$					
e.I.1(n)	$(-2, -3, -1;$ $2n-2, 3n, n-4; 6n)$	$(-\frac{1}{3n}, -\frac{n+1}{3n}; -\frac{1}{2n}, -\frac{n+1}{2n};$ $-\frac{1}{6n}, -\frac{n+1}{6n})$	$(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{6})$	$(3n+3, 3; 2n+2, 2;$ $6n+6, 6)$	e.II(n) via 3_c e.IV($n, 1, \infty$) via 2
e.I.2(n)	$(-1, -2, -1;$ $n-2, 2n, n-2; 4n)$	$(-\frac{1}{4n}, -\frac{n+1}{4n}; -\frac{1}{2n}, -\frac{n+1}{2n};$ $-\frac{1}{4n}, -\frac{n+1}{4n})$	$(-\frac{1}{4}, -\frac{1}{2}, -\frac{1}{4})$	$(4n+4, 4; 2n+2, 2;$ $4n+4, 4)$	
e.II(n)	$(-1, -1, -1;$ $n-1, n-1, n-1; 3n)$	$(-\frac{1}{3n}, -\frac{n+1}{3n}; -\frac{1}{3n}, -\frac{n+1}{3n};$ $-\frac{1}{3n}, -\frac{n+1}{3n})$	$(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$	$(3n+3, 3; 3n+3, 3;$ $3n+3, 3)$	
e.IV.4(n)	$(-1, 0, -1;$ $n, -2, n; 2n)$	$(-\frac{1}{3n}, -\frac{n+1}{3n}; 0, 0;$ $-\frac{1}{2n}, -\frac{n+1}{2n})$	$(-\frac{1}{2}, 0, -\frac{1}{2})$	$(2n+2, 2; \infty, \infty;$ $2n+2, 2)$	
e.IV($n, 1, \infty$)	$(0, -1, 0;$ $-1, n+1, -1; n)$	$(0, 0; -\frac{1}{n}, -\frac{n+1}{n};$ $0, 0)$	$(0, -1, 0)$	$(\infty, \infty; n+1, 1;$ $\infty, \infty)$	
e.IV($n, 1, r$)	$(n+1, r(n+1), -1;$ $-1, -1, n+1; n)$	$(-\frac{n+1}{nr}, -\frac{1}{nr}; -\frac{n+1}{n}, -\frac{1}{n};$ $\frac{1}{nr}, \frac{n+1}{nr})$	$(\frac{1}{r}, 1, \frac{1}{r})$	$(r, r(n+1); 1, n+1;$ $-r(n+1), -r)$	
$n = 1$ (i.e., $\bar{n} = 2$)					
I	$(2, 2, 1; -1, -1, 2; 3)$	$(-2, 0; -2, 0; -1, 2)$	$(2, 2, 3)$	$(1, \infty; 1, \infty; 2, -1)$	I via 2
I.1	$(2, 4, 1; -3, 1, 2; 3)$	$(-1, -\frac{1}{2}; -2, 0; -\frac{1}{2}, 1)$	$(\frac{1}{2}, 2, \frac{3}{2})$	$(2, 4; 1, \infty; 4, -2)$	
II	$(1, 1, 0; 0, 0, 0; 1)$	$(-2, 0; -2, 0; 0, 1)$	$(2, 2, 1)$	$(1, \infty; 1, \infty; \infty, -2)$	II via 2
II.1	$(1, 2, 0; -1, 1, 0; 1)$	$(-1, -\frac{1}{2}; -2, 0; 0, \frac{1}{2})$	$(\frac{1}{2}, 2, \frac{1}{2})$	$(2, 4; 1, \infty; \infty, -4)$	
III	$(1, 1, 1; 0, -1, 1; 2)$	$(-2, 1; -2, 0; -2, 2)$	$(3, 2, 4)$	$(1, -2; 1, \infty; 1, -1)$	
IV	$(2, 2, 1; 1, -1, 0; 3)$	$(-2, 1; -2, 0; -1, 1)$	$(3, 2, 2)$	$(1, -2; 1, \infty; 2, -2)$	
V	$(1, 1, 1; 0, 0, 0; 2)$	$(-2, 1; -2, 1; -2, 1)$	$(3, 3, 3)$	$(1, -2; 1, -2; 1, -2)$	V via 2
V.1	$(1, 2, 1; -2, 2, 0; 2)$	$(-1, -\frac{1}{2}; -2, 1; -1, \frac{1}{2})$	$(\frac{1}{2}, 3, \frac{3}{2})$	$(2, 4; 1, -2; 2, -4)$	

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system	$(a_1, a_2, a_3; b_1, b_2, b_3; c)$	$(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$	$(\alpha_1, \alpha_2, \alpha_3)$	$(r_1, r'_1; r_2, r'_2; r_3, r'_3)$	preimage(s), [notes]
V.2	$(3, 2, 3; -2, -2, 4; 2)$	$(-1, -\frac{1}{2}; -\frac{2}{3}, -\frac{1}{3}; -1, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$	$(2, 4; 3, 6; 2, -4)$	V.1,3;V via 3, 2; 6_c [Ch-XI(4) for x_3] V via 3_c
V.3	$(1, 3, 1; -2, 4, -2; 2)$	$(-\frac{2}{3}, -\frac{1}{3}; -2, 1; -\frac{2}{3}, -\frac{1}{3})$	$(\frac{1}{3}, 3, \frac{1}{3})$	$(3, 6; 1, -2; 3, 6)$	VI via 2
VI	$(1, 2, 2; -2, 1, 1; 3)$	$(-1, 0; -2, 1; -2, 1)$	$(1, 3, 3)$	$(2, \infty; 1, -2; 1, -2)$	
VI.1	$(2, 4, 1; -3, 5, -2; 3)$	$(-1, -\frac{1}{2}; -2, 1; -\frac{1}{2}, 0)$	$(\frac{1}{2}, 3, \frac{1}{2})$	$(2, 4; 1, -2; 4, \infty)$	
VII.1	$(1, 2, 1; -1, 0, 1; 2)$	$(-1, 0; -2, 0; -1, 1)$	$(1, 2, 2)$	$(2, \infty; 1, \infty; 2, -2)$	
VII.2	$(1, 3, 1; -2, 1, 1; 2)$	$(-\frac{2}{3}, -\frac{1}{3}; -2, 0; -\frac{2}{3}, \frac{2}{3})$	$(\frac{1}{3}, 2, \frac{4}{3})$	$(3, 6; 1, \infty; 3, -3)$	
VIII.1	$(2, 6, 1; -3, 3, 0; 3)$	$(-\frac{2}{3}, -\frac{1}{3}; -2, 0; -\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, 2, \frac{2}{3})$	$(3, 6; 1, \infty; 6, -6)$	
X.1	$(1, 4, 2; -2, 1, 1; 3)$	$(-\frac{1}{2}, 0; -2, 0; -1, \frac{1}{2})$	$(\frac{1}{2}, 2, \frac{3}{2})$	$(4, \infty; 1, \infty; 2, -4)$	
XI.1	$(2, 6, 1; -1, 3, -2; 3)$	$(-\frac{2}{3}, 0; -2, 0; -\frac{1}{3}, 0)$	$(\frac{2}{3}, 2, \frac{1}{3})$	$(3, \infty; 1, \infty; 6, \infty)$	
XII.1	$(1, 4, 1; -2, 2, 0; 2)$	$(-\frac{1}{2}, -\frac{1}{4}; -2, 0; -\frac{1}{2}, \frac{1}{4})$	$(\frac{1}{2}, 2, \frac{3}{4})$	$(4, 8; 1, \infty; 4, -8)$	
XIII.1	$(1, 4, 1; -1, 2, -1; 2)$	$(-\frac{1}{2}, 0; -2, 0; -\frac{1}{2}, 0)$	$(\frac{1}{2}, 2, \frac{1}{2})$	$(4, \infty; 1, \infty; 4, \infty)$	
XIII.2	$(6, 4, 3; -1, -1, 2; 1)$	$(-1, -\frac{1}{2}; -\frac{2}{3}, -\frac{1}{3}; -\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$	$(2, 4; 3, 6; 4, \infty)$	
XIII.3	$(2, 2, 1; -1, 1, 0; 1)$	$(-1, -\frac{1}{2}; -1, 0; -\frac{1}{2}, 0)$	$(\frac{1}{2}, 1, \frac{1}{2})$	$(2, 4; 2, \infty; 4, \infty)$	
XIII.4	$(1, 1, 1; 0, 0, 0; 1)$	$(-1, 0; -1, 0; -1, 0)$	$(1, 1, 1)$	$(2, \infty; 2, \infty; 2, \infty)$	
XIII.5	$(2, 3, 2; -1, 2, -1; 1)$	$(-\frac{2}{3}, -\frac{1}{3}; -1, 0; -\frac{2}{3}, -\frac{1}{3})$	$(\frac{1}{3}, 1, \frac{1}{3})$	$(3, 6; 2, \infty; 3, 6)$	
e.IV(1,2, r)	$(2, 2r, 0; 0, -2, 2; 2)$	$(-\frac{2}{r}, 0; -2, -1; 0, \frac{2}{r})$	$(\frac{2}{r}, 1, \frac{2}{r})$	$(r, \infty; 1, 2; \infty, -r)$	e.IV(1,4, r) via 2
e.IV(1,3, r)	$(2, 2r, 1; 1, -3, 2; 3)$	$(-\frac{2}{r}, \frac{1}{r}; -2, -1; -\frac{1}{r}, \frac{2}{r})$	$(\frac{2}{r}, 1, \frac{2}{r})$	$(r, -2r; 1, 2; 2r, -r)$	
e.IV(1,4, r)	$(2, 2r, 2; 2, -4, 2; 4)$	$(-\frac{2}{r}, \frac{2}{r}; -2, -1; -\frac{2}{r}, \frac{2}{r})$	$(\frac{4}{r}, 1, \frac{4}{r})$	$(r, -r; 1, 2; r, -r)$	
e.IV.1(r)	$(r, 2, r; -2, 4, -2; 2)$	$(-1, -\frac{1}{2}; -\frac{2}{r}, \frac{2}{r}; -1, -\frac{1}{2})$	$(\frac{1}{2}, \frac{4}{r}, \frac{1}{2})$	$(2, 4; r, -r; 2, 4)$	

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system	$(a_1, a_2, a_3; b_1, b_2, b_3; c)$	$(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$	$(\alpha_1, \alpha_2, \alpha_3)$	$(r_1, r'_1; r_2, r'_2; r_3, r'_3)$	preimage(s), [notes]
$n = 2$ (i.e., $\bar{n} = 3/2$)					
I	$(1, 1, 1; 0, 0, 1; 2)$	$(-\frac{3}{2}, \frac{1}{2}; -\frac{3}{2}, \frac{1}{2}; -\frac{3}{2}, \frac{3}{2})$	$(2, 2, 3)$	$(1, -3; 1, -3; 1, -1)$	I via 2
I.1	$(1, 2, 1; -1, 1, 1; 2)$	$(-\frac{3}{4}, -\frac{1}{4}; -\frac{3}{2}, \frac{1}{2}; -\frac{3}{4}, \frac{3}{4})$	$(\frac{1}{2}, 2, \frac{3}{2})$	$(2, 6; 1, -3; 2, -2)$	
II	$(3, 3, 1; 1, 1, 0; 4)$	$(-\frac{3}{2}, \frac{1}{2}; -\frac{3}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2})$	$(2, 2, 1)$	$(1, -3; 1, -3; 3, -3)$	II via 2 [Ch-XI(9) for x_1]
II.1	$(3, 6, 1; -2, 4, 0; 4)$	$(-\frac{3}{4}, -\frac{1}{4}; -\frac{3}{2}, \frac{1}{2}; -\frac{1}{4}, \frac{1}{4})$	$(\frac{1}{2}, 2, \frac{1}{2})$	$(2, 6; 1, -3; 6, -6)$	
III.1	$(4, 6, 4; 3, -1, -1; 2)$	$(-\frac{1}{2}, \frac{1}{6}; -\frac{3}{4}, -\frac{1}{4}; -\frac{1}{2}, -\frac{1}{6})$	$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	$(3, -9; 2, 6; 3, 9)$	
III.2	$(1, 3, 1; -1, 2, 0; 2)$	$(-\frac{1}{2}, -\frac{1}{6}; -\frac{3}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{6})$	$(\frac{1}{3}, 2, \frac{2}{3})$	$(3, 9; 1, -3; 3, -9)$	
e.IV(2,2, r)	$(3, 3r, 1; 1, -2, 3; 4)$	$(-\frac{3}{2r}, \frac{1}{2r}; -\frac{3}{2}, -\frac{1}{2}; -\frac{1}{2r}, \frac{3}{2r})$	$(\frac{2}{r}, 1, \frac{2}{r})$	$(r, -3r; 1, 3; 3r, -r)$	e.IV(2,3, r) via 2
e.IV(2,3, r)	$(3, 3r, 3; 3, -3, 3; 6)$	$(-\frac{3}{2r}, \frac{3}{2r}; -\frac{3}{2}, -\frac{1}{2}; -\frac{3}{2r}, \frac{3}{2r})$	$(\frac{3}{r}, 1, \frac{3}{r})$	$(r, -r; 1, 3; r, -r)$	
e.IV.2(r)	$(r, 2, r; -1, 3, -1; 2)$	$(-\frac{3}{4}, -\frac{1}{4}; -\frac{3}{2r}, \frac{3}{2r}; -\frac{3}{4}, -\frac{1}{4})$	$(\frac{1}{2}, \frac{3}{r}, \frac{1}{2})$	$(2, 6; r, -r; 2, 6)$	
$n = 3$ (i.e., $\bar{n} = 4/3$)					
I	$(2, 2, 1; 1, 1, 0; 3)$	$(-\frac{4}{3}, \frac{2}{3}; -\frac{4}{3}, \frac{2}{3}; -\frac{2}{3}, \frac{1}{3})$	$(2, 2, 1)$	$(1, -2; 1, -2; 2, -4)$	I via 2
I.1	$(2, 4, 1; -1, 3, 0; 3)$	$(-\frac{4}{3}, -\frac{1}{6}; -\frac{4}{3}, \frac{2}{3}; -\frac{1}{3}, \frac{1}{6})$	$(\frac{1}{2}, 2, \frac{1}{2})$	$(2, 8; 1, -2; 4, -8)$	
e.IV(3,2, r)	$(4, 4r, 2; 2, -2, 4; 6)$	$(-\frac{4}{3r}, \frac{2}{3r}; -\frac{4}{3}, -\frac{1}{3}; -\frac{2}{3r}, \frac{4}{3r})$	$(\frac{2}{r}, 1, \frac{2}{r})$	$(r, -2r; 1, 4; 2r, -r)$	
$n = 5$ (i.e., $\bar{n} = 6/5$)					
e.IV(5,2, r)	$(6, 6r, 4; 4, -2, 6; 10)$	$(-\frac{6}{5r}, \frac{4}{5r}; -\frac{6}{5}, -\frac{1}{5}; -\frac{4}{5r}, \frac{6}{5r})$	$(\frac{2}{r}, 1, \frac{2}{r})$	$(r, -\frac{3}{2}r; 1, 6; \frac{3}{2}r, -r)$	[N.B.: even r only]
$n = \infty$ (i.e., $\bar{n} = 1$)					
e.I.1(∞)	$(0, 0, 0; 2, 3, 1; 6)$	$(0, -\frac{1}{3}; 0, -\frac{1}{2}; 0, -\frac{1}{6})$	$(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{6})$	$(\infty, 3; \infty, 2; \infty, 6)$	e.II(∞) via 3_c e.IV($\infty, 1, \infty$) via 2
e.I.2(∞)	$(0, 0, 0; 1, 2, 1; 4)$	$(0, -\frac{1}{4}; 0, -\frac{1}{2}; 0, -\frac{1}{4})$	$(-\frac{1}{4}, -\frac{1}{2}, -\frac{1}{4})$	$(\infty, 4; \infty, 2; \infty, 4)$	
e.II(∞)	$(0, 0, 0; 1, 1, 1; 3)$	$(0, -\frac{1}{3}; 0, -\frac{1}{3}; 0, -\frac{1}{3})$	$(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$	$(\infty, 3; \infty, 3; \infty, 3)$	
e.IV.4(∞)	$(0, 0, 0; 1, 0, 1; 2)$	$(0, -\frac{1}{2}; 0, 0; 0, -\frac{1}{2})$	$(-\frac{1}{2}, 0, -\frac{1}{2})$	$(\infty, 2; \infty, \infty; \infty, 2)$	
e.IV($\infty, 1, \infty$)	$(0, 0, 0; 0, 1, 0; 1)$	$(0, 0; 0, -1; 0, 0)$	$(0, -1, 0)$	$(\infty, \infty; \infty, 1; \infty, \infty)$	
e.IV($\infty, 1, r$)	$(1, r, 0; 0, 0, 1; 1)$	$(-\frac{1}{r}, 0; -1, 0; 0, \frac{1}{r})$	$(\frac{1}{r}, 1, \frac{1}{r})$	$(r, \infty; 1, \infty; \infty, -r)$	e.IV($\infty, 2, r$) via 2
e.IV($\infty, 2, r$)	$(1, r, 1; 1, 0, 1; 2)$	$(-\frac{1}{r}, \frac{1}{r}; -1, 0; -\frac{1}{r}, \frac{1}{r})$	$(\frac{1}{r}, 1, \frac{1}{r})$	$(r, -r; 1, \infty; r, -r)$	
e.IV.3(r)	$(r, 2, r; 0, 2, 0; 2)$	$(-\frac{1}{2}, 0; -\frac{1}{r}, \frac{1}{r}; -\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{2}{r}, \frac{1}{2})$	$(2, \infty; r, -r; 2, \infty)$	

solution $t = t(\tau)$ of the gSE near any of its singular points, say $\tau = \tau_*$. For the PP to obtain, the solution must not be branched.

As was mentioned at the end of § 2.2, by substituting into the gSE the formal statement $t = t(\tau) \sim t_* + C(\tau - \tau_*)^p$ where $t_* \neq t_1, t_2, t_3$, it is easy to deduce that $p \in \{\pm 1, \pm(n+1)\}$, with the statement holding as $\tau \rightarrow \tau_*$ if $\operatorname{Re} p > 0$ and as $\tau \rightarrow \infty$ if $\operatorname{Re} p < 0$. Here $n := 1/(\bar{n} - 1)$ and $\bar{n} = (n + 1)/n$. Formally, this is the source of the restriction that n be an integer or ∞ .

If instead $t_* = t_i$ for one of $i = 1, 2, 3$, one still finds that a leading exponent p governing $\tau \rightarrow \infty$ behavior (where $\operatorname{Re} p < 0$) must satisfy $p \in \{-1, -n - 1\}$; but now, an exponent p governing $\tau \rightarrow \tau_*$ behavior (where $\operatorname{Re} p > 0$) must satisfy $p \in \{r_i, r'_i\}$, where $(r_i, r'_i) := -\bar{n}(1/\nu_i, 1/\nu'_i)$. The appearance of new exponents is a reflection of the following. In Papperitz-based integration (with the local function $\tau = \tau(t)$ and its inverse $t = t(\tau)$ defined as in Theorem 2.1), the two Frobenius solutions of the PE at $t = t_i$, say $f^{[i]}, f^{[i']}$, turn out to yield gSE solutions $t = t^{[i]}(\tau)$, $t^{[i']}(\tau)$ with formal asymptotic behavior

$$(4.1a) \quad t^{[i]}(\tau) \sim t_i + \text{const} \times (\tau - \tau_*)^{r_i},$$

$$(4.1b) \quad t^{[i']}(\tau) \sim t_i + \text{const} \times (\tau - \tau_*)^{r'_i},$$

where $(r_i, r'_i) := -\bar{n}(1/\nu_i, 1/\nu'_i)$, and lower-order terms with exponents differing by integers from r_i , resp. r'_i , are omitted. In either approach, one expects that for the gSE to have the PP, r_i, r'_i must be integers (at least, if they are finite, with positive real parts). On each line of Table 6, $(r_1, r'_1; r_2, r'_2; r_3, r'_3)$ is given. The identity

$$(4.2) \quad -(n+1)^{-1} + \sum_{i=1}^3 (r_i^{-1} + r'_i{}^{-1}) = 1$$

is a restatement of the Fuchsian condition in terms of these exponents.

The PE having a 2-dimensional space of solutions, each of its basis functions $f^{[i]}, f^{[i']}$ can be continuously deformed; and one finds that from any nontrivial mixture comes a gSE solution $t = t(\tau)$ with one of two formal behaviors:

$$(4.3a) \quad \bar{t}^{[i]}(\tau) \sim t_i + \text{const} \times (\tau - \tau_*)^{r_i} \times [1 + \text{const} \times (\tau - \tau_*)^{q_i}],$$

$$(4.3b) \quad \bar{t}^{[i']}(\tau) \sim t_i + \text{const} \times (\tau - \tau_*)^{r'_i} \times [1 + \text{const} \times (\tau - \tau_*)^{q'_i}],$$

where lower-order terms are omitted, and the definition of the auxiliary exponents is $(q_i, q'_i) := \bar{n}((\nu_i - \nu'_i)/\nu_i, (\nu'_i - \nu_i)/\nu'_i)$, so that if defined they satisfy

$$(4.4) \quad (n+1)^{-1} + q_i^{-1} + q'_i{}^{-1} = 1, \quad i = 1, 2, 3.$$

The ‘mixed’ solution (4.3a), resp. (4.3b), occurs if $\nu_i < \nu'_i$, resp. $\nu'_i < \nu_i$. The conditions that for $i = 1, 2, 3$, each of $r_i, r'_i; q_i, q'_i$ be an integer (or ∞ or undefined [i.e. ‘0/0’], either indicating that there is no singularity at all), imposed along with the condition that $n := 1/(\bar{n} - 1)$ be an integer or ∞ , and the Fuchsian condition (4.2), greatly restrict the possible $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$.

Formally, this is the origin of the list of parameter vectors in Table 6. The tables of [12] give r_i and q_i (or rather the equivalent quantity $p_i := q_i - 1$) for $i = 1, 2, 3$, which permits the vectors to be reconstructed. The labeling scheme used in Table 6 is that of [12], though explicit parametrizations (by (n) , (r) or (n, q, r) , as appropriate) have been added. For consistency the family e.III(n, r) of [12] has been relabeled e.IV($n, 1, r$). In parametrized families the parameter q is a positive integer,

r is a nonzero integer, and as stated $n := 1/(\bar{n} - 1)$, if finite, can take any integral value (except for $n = 0, -1$, which would imply $\bar{n} = 0, \infty$, which are improper values; and for $n = -2$, which would yield the DH [and SE] value $\bar{n} = 1/2$).

Of special note are the five parametrized families e.I.1(n), e.I.2(n), e.II(n), e.IV.4(n), and e.IV($n, 1, \infty$). Each has been defined so that it is ‘pseudo-Euclidean,’ in that the angular parameters $\alpha_1, \alpha_2, \alpha_3$ satisfy $\alpha_1 + \alpha_2 + \alpha_3 = -1$. Because of this convention, the last of these, the family e.IV($n, 1, \infty$), is not the formal $r \rightarrow \infty$ limit of the separately listed family e.IV($n, 1, r$). Its parameters ν_2, ν'_2 have been transposed, so that $(\alpha_1, \alpha_2, \alpha_3)$ equals $(0, -1, 0)$ rather than $(0, 1, 0)$. That is, the element $[2_-] \in \mathfrak{G}$ has been applied. Without this transposition the alternative parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n})$ would be improper in the sense of Definition 2.2, because $\rho^{-1} := (1 - \alpha_1 - \alpha_2 - \alpha_3)/2$, where $\alpha_i := \nu'_i - \nu_i$, would equal zero; and according to Eq. (2.14), $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ would not be defined. (The gSE (2.41) is unaffected by the transposition, but without it Theorem 2.9 would not apply: there would be no gDH \leftrightarrow gSE correspondence.)

The reader may wonder why the HQDS $\dot{x}_i = x_j x_k$ (known as the 3-dimensional Nahm system, a variant of the Euler–Poincaré top), which is gDH(0, 0, 0; 0, 0, 0; 1) and is known to be completely integrable [22] and to have the Painlevé property, is not listed in Table 6. In fact it is present in disguise. It is one of the systems on the \mathfrak{G} -orbit of the $n = 1$ system XIII.4, which is gDH(1, 1, 1; 0, 0, 0; 1). The two gDH systems are linearly equivalent, being related by the involution $[1_-][2_-][3_-] \in \mathfrak{G}$, i.e. by $\nu_i \leftrightarrow \nu'_i$ ($\forall i$), as one sees from the formulas of Proposition 3.8.

For any HQDS, Painlevé and integrability properties can be explored with the aid of Kovalevskaya exponents [27, 67], which provide information on the linearization of the dynamics specified by $\dot{x} = Q(x)$ around each ray solution. If the non-associative algebra \mathfrak{A} for a d -dimensional HQDS has an idempotent p with accompanying ray solution $x(\tau) = -(\tau - \tau_*)^{-1}p$, the Kovalevskaya exponents associated to p are defined as the eigenvalues of the $d \times d$ matrix $I - [\partial Q_i / \partial x_j]_{ij}(x = p)$. A necessary condition for an HQDS to be algebraically integrable is that each Kovalevskaya exponent be rational [67]; and by definition, each must be an integer if the HQDS is to have the PP. An easy calculation applied to the gDH system (1.4) yields the following.

Lemma 4.2. *The Kovalevskaya exponents associated to the seven canonical idempotents $p = p_0, p_1, p_2, p_3, p'_1, p'_2, p'_3 \in \mathfrak{A}$ of any (generic) gDH system that are defined in Proposition 2.3, i.e., associated to the seven ray solutions along these idempotents, are respectively*

$$\begin{aligned} \mathcal{R}_0 &= \{-1, \bar{n}/(\bar{n} - 1), \bar{n}/(\bar{n} - 1)\} =: \{-1, n + 1, n + 1\}, \\ \mathcal{R}_i &= \{-1, r_i, q_i\}, & i = 1, 2, 3, \\ \mathcal{R}'_i &= \{-1, r'_i, q'_i\}, & i = 1, 2, 3, \end{aligned}$$

where $(r_i, r'_i) = -\bar{n}(1/\nu_i, 1/\nu'_i)$ and $(q_i, q'_i) = \bar{n}((\nu_i - \nu'_i)/\nu_i, (\nu'_i - \nu_i)/\nu'_i)$, as above; so that if defined, they satisfy Eqs. (4.2), (4.4). In each of \mathcal{R}_i and \mathcal{R}'_i , the eigendirection corresponding to the first of the three exponents (i.e., -1) lies along p , and that of the third (i.e., q_i , resp. q'_i) lies in the $x_j - x_k = 0$ subspace.

If for any of the seven, any Kovalevskaya exponent is formally infinite or undefined (i.e. ‘0/0’), it indicates nongenericity: the idempotent is absent, and the ray solution degenerates to a ray of constant solutions (i.e., nilpotents).

TABLE 7. ODEs (i.e., algebraic curves in t, \dot{t}) determining the solutions of many gDH systems with the Painlevé property. (For e.IV(n, q, r) systems, t is represented as u^r .)

system	ODE	type of curve
n an arbitrary integer ($\neq 0, -1, -2$), with $\bar{n} = (n+1)/n$		
e.IV($n, 1, r$)	$\dot{u}^{n+1} = (K_1 u - K_2)^n$	rational
$n = 1$ (i.e., $\bar{n} = 2$)		
I	$\dot{t}^2 = K_1 t^2(2t - 3) + K_2$	elliptic; rational if $K_1 K_2(K_1 - K_2) = 0$
II	$\dot{t}^2 = K_1 t^2 - K_2(2t - 1)$	rational
III	$\dot{t}^2 = K_1 t^3(3t - 4) + K_2$	elliptic; rational if $K_1 K_2(K_1 - K_2) = 0$
IV	$\dot{t}^2 = K_1 t^3 - K_2(3t - 2)$	elliptic; rational if $K_1 K_2(K_1 - K_2) = 0$
V	$\dot{t}^2 = K_1 t^3(t - 2) + K_2(2t - 1)$	elliptic ($j = 0$); rational if $K_1 K_2(K_1 - K_2) = 0$
VI	$\dot{t}^2 = K_1 t^2(t^2 - 3t + 3) - K_2 t$	elliptic; rational if $K_1 K_2(K_1 - K_2) = 0$
e.IV(1, 2, r)	$\dot{u}^2 = K_1 u^2 - K_2$	rational
e.IV(1, 3, r)	$\dot{u}^2 = K_1 u^3 - K_2$	elliptic ($j = 0$); rational if $K_1 K_2 = 0$
e.IV(1, 4, r)	$\dot{u}^2 = K_1 u^4 - K_2$	elliptic ($j = 12^3$); rational if $K_1 K_2 = 0$
$n = 2$ (i.e., $\bar{n} = 3/2$)		
I	$\dot{t}^3 = [K_1 t^2(2t - 3) + K_2]^2$	elliptic ($j = 0$); rational if $K_1 K_2(K_1 - K_2) = 0$
II	$\dot{t}^3 = [K_1 t^2 - K_2(2t - 1)]^2$	elliptic ($j = 0$); rational if $K_1 K_2(K_1 - K_2) = 0$
e.IV(2, 2, r)	$\dot{u}^3 = (K_1 u^2 - K_2)^2$	elliptic ($j = 0$); rational if $K_1 K_2 = 0$
e.IV(2, 3, r)	$\dot{u}^3 = (K_1 u^3 - K_2)^2$	elliptic ($j = 0$); rational if $K_1 K_2 = 0$
$n = 3$ (i.e., $\bar{n} = 4/3$)		
I	$\dot{t}^4 = [K_1 t^2 - K_2(2t - 1)]^3$	elliptic ($j = 12^3$); rational if $K_1 K_2(K_1 - K_2) = 0$
e.IV(3, 2, r)	$\dot{u}^4 = (K_1 u^2 - K_2)^3$	elliptic ($j = 12^3$); rational if $K_1 K_2 = 0$
$n = 5$ (i.e., $\bar{n} = 6/5$)		
e.IV(5, 2, r)	$\dot{u}^6 = (K_1 u^2 - K_2)^5$	hyperelliptic, genus = 2; rational if $K_1 K_2 = 0$
$n = \infty$ (i.e., $\bar{n} = 1$)		
e.IV($\infty, 1, r$)	$\dot{u} = K_1 u - K_2$	rational
e.IV($\infty, 2, r$)	$\dot{u} = K_1 u^2 - K_2$	rational

Example. Any proper DH system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$, which is a (proper) gDH system with $(\nu_i, \nu'_i) = (-\alpha_i/2, \alpha_i/2)$ and $\bar{n} = 1/2$, has Kovalevskaya exponents $\mathcal{R}_0 = \{-1, -1, -1\}$ and $\mathcal{R}_i = \{-1, 1/\alpha_i, 1\}$, $\mathcal{R}'_i = \{-1, -1/\alpha_i, 1\}$, for $i = 1, 2, 3$. If $\alpha_i = 0$ for any i then $\mathcal{R}_i, \mathcal{R}'_i$ will formally include exponents that are infinite or undefined, which simply indicates that $\mathcal{R}_i, \mathcal{R}'_i$ should be omitted: though each element $e = e_i, e'_i$ satisfies $e * e \propto e$, it is a nilpotent ($e * e = 0$), and hence there is no idempotent $p \propto e$, and no ray solution $x(\tau) = -(\tau - \tau_*)^{-1}p$.

Thus the original Darboux system $\text{DH}(0, 0, 0 | c)$ has as its only ray solution $x(\tau) = -(2/c)(\tau - \tau_*)^{-1}(1, 1, 1)$, i.e. the ray solution (1.6) with all components coincident, which lies along the multiplicative identity element $p_0 = (2/c)e_0 = (2/c)(e_1 + e_2 + e_3)$. Its set of Kovalevskaya exponents \mathcal{R}_0 is $\{-1, -1, -1\}$.

It follows from the lemma that conditions for the PP to obtain are that $n := 1/(\bar{n} - 1)$ be a nonzero integer or ∞ , and that each of $r_i, r'_i; q_i, q'_i$, $i = 1, 2, 3$, be an integer or ∞ , if defined. Thus, similar necessary conditions emerge from a Kovalevskaya–Painlevé analysis as from the proper gDH \leftrightarrow gSE correspondence.

The reader may have noticed a small lacuna in the proof of Theorem 4.1. As stated, the theorem is not an immediate corollary of Carton-LeBrun’s classification of non-SE gSE’s that have the PP. This is because it does not restrict the *type* of solution $x = x(\tau)$ that cannot have a (movable) branch point. The gDH \leftrightarrow gSE correspondence of Theorem 2.9 applies only to *noncoincident* solutions, with no pair of components coinciding (necessarily, at all τ). To rule out branch points in any *coincident* solution of each of the gDH’s listed in Table 6, an auxiliary argument is needed.

Suppose that two components coincide; say, $x_j \equiv x_k$. By substituting this into the gDH system (1.4) one obtains a 2-dimensional HQDS for (x_i, x_j) . Such systems have long been classified [50], and their integrability and Painlevé properties are known. It is known in particular that for a 2-dimensional HQDS, integer Kovalevskaya exponents are sufficient for the PP. But, the Kovalevskaya exponents of this reduced 2-dimensional HQDS must be integers, since they are a subset of the exponents of the original gDH system: they come from $p = p_0, p_i, p'_i$, each of which lies in the $x_j - x_k = 0$ subspace, and in particular they are the two eigenvalues in each of $\mathcal{R}_0, \mathcal{R}_i, \mathcal{R}'_i$ with eigendirections in that subspace. That is, they are $\{-1, n + 1\}$, $\{-1, q_i\}$, $\{-1, q'_i\}$. (If any of these three pairs contains a formally infinite or undefined exponent, indicating nilpotence rather than idempotence, it must be omitted.) This concludes the proof of Theorem 4.1.

There is much to say about the non-gDH systems with the PP in Table 6. Strikingly, many are related by rational morphisms, i.e. by rational solution-preserving maps $x = \Phi(\tilde{x})$. The maps, which include **2, 3, 3_c, 6, 6_c**, are shown in the final column. (Permutations of components x_i and \tilde{x}_i , if a part of the morphism, are not shown.) The labeling scheme indicates whether any gDH system in the table is obtained in this way from a ‘base’ system. For instance, the $n = 1$ (i.e., $\bar{n} = 2$) systems V.1, V.2, V.3 are images of the system V by **2, 6_c, 3_c**. Moreover V.2 is the image of V.1 by **3**, and of V.3 by **2**. Thus, this quadruple of gDH systems with the PP illustrates the **6_c \sim 3 \circ 2 \sim 2 \circ 3_c** ‘diamond’ in Figure 1!

For some ‘image’ gDH systems in the table, no base system is given. (For instance, the $n = 2$ systems include III.1 and III.2, but no III.) In each such case, the gSE associated to the system is the image under a rational map $t = R(\tilde{t})$ of a nonlinear third-order ODE that is not of the ‘triangular’ gSE form (2.41), having

more than three singular values on \mathbb{P}_t^1 . Equivalently, the associated PE on \mathbb{P}_t^1 can be lifted along $t = R(\tilde{t})$ to a linear Fuchsian ODE on $\mathbb{P}_{\tilde{t}}^1$ that has more than three singular points, and is therefore not a PE. Generalized Schwarzian equations with more than three singular points can be found in the tables of [12], but are a bit beyond the scope of the present paper.

Each of the base systems in Table 6, or more precisely the corresponding gSE's, can be integrated [12, 24]. PE-based integration suffices. In the notation of Theorem 2.1, let $\tilde{t} = K^2(t)f^{1/\bar{n}}(t)$, where $\bar{n} = (n+1)/n$ and f is a nonzero solution of the associated PE (2.1b), with exponents $(\mu_1, \mu'_1; \mu_2, \mu'_2; \mu_3, \mu'_3)$ computed from $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3)$ and the offset vector κ by (2.15). But (see the Remark after that theorem), $K \equiv \text{const}$ when $(t_1, t_2, t_3) = (0, 1, \infty)$ and $\kappa = (0, 0, 1)$. In this case one can write $\tilde{t}^{n+1} \propto f^n(t)$, where $f = K_1 f^{(1)} + K_2 f^{(2)}$ is the general solution of the PE. This shows that the space of possible $t = t(\tau)$, which are solutions of the gSE (2.41), is closed under affine transformations $\tau \mapsto A\tau + B$, as one expects. For nearly all base systems in the table the PE turns out to have trivial monodromy, which implies that the PE solutions $f^{(1)}(t), f^{(2)}(t)$ are polynomials. If this is the case then t, \tilde{t} are algebraically related: they are functions on an algebraic curve.

Theorem 4.3. *Of the proper gDH systems with the PP listed in Table 6, the base systems are integrated as follows. For each, a first-order ODE for the gSE solution $t = t(\tau)$ (i.e., an algebraic curve in t, \tilde{t}), where $t = -(x_2 - x_3)/(x_1 - x_2)$, is given in Table 7. The parameters $K_1, K_2 \in \mathbb{C}$ are free, and after $t = t(\tau)$ is obtained, the corresponding noncoincident gDH solution $x = x(\tau)$ is computed from the $t(\cdot) \mapsto x(\cdot)$ map (2.32).*

The five pseudo-Euclidean families at the head of Table 6, namely e.I.1(n), e.I.2(n), e.II(n), e.IV.4(n) and e.IV(n,1, ∞), omitted from Table 7 because the associated PE's have nonpolynomial solutions, are integrated thus. Let

$$\sigma_C(\tau) := \begin{cases} C + \tau^{n+1}, & n \neq \infty, \\ C + e^\tau, & n = \infty, \end{cases}$$

where $C \in \mathbb{C}$ is a free parameter. Then up to an affine transformation $\tau \mapsto A\tau + B$, the general solution $t = t(\tau)$ of the gSE associated to a gDH system in any of them is $t = \psi(\sigma_C(\tau))$, where $\psi = \psi(\sigma)$ is any nonconstant function satisfying

$$\begin{cases} (d\psi/d\sigma)^6 = \psi^4(\psi - 1)^3, & \text{for e.I.1(n);} \\ (d\psi/d\sigma)^4 = \psi^3(\psi - 1)^2, & \text{for e.I.2(n);} \\ (d\psi/d\sigma)^3 = \psi^2(\psi - 1)^2, & \text{for e.II(n);} \\ (d\psi/d\sigma)^2 = \psi^1(\psi - 1)^2, & \text{for e.IV.4(n);} \\ (d\psi/d\sigma)^1 = \psi^1(\psi - 1)^0, & \text{for e.IV(n,1,\infty).} \end{cases}$$

There is also a special solution that can be chosen to be $t = \psi(\tau)$.

Remark. For each of the five pseudo-Euclidean families, the curve in $\psi, d\psi/d\sigma$ is of the form $d\psi/d\sigma = \psi^{\alpha_1+1}(\psi - 1)^{\alpha_2+1}$. The family e.IV.4(n) is included in this theorem for completeness, but is not a family of base systems. As is indicated in Table 6, e.IV.4(n) comes from e.IV(n,1, ∞) via **2**. One can simply choose

$$(4.5) \quad \psi(\sigma) = \left(\frac{e^\sigma + 1}{e^\sigma - 1} \right)^2, \quad \text{resp.} \quad e^\sigma,$$

for e.IV.4(n), resp. e.IV($n, 1, \infty$).

Sketch of Proof. The curves $\dot{t}^{n+1} = (K_1 f^{(1)} + K_2 f^{(2)})^n$ in Table 7 are adapted from [12, Tab. VIII], with modifications to ensure consistency. For each system in Table 7, $K_1 f^{(1)} + K_2 f^{(2)}$ is a solution of the associated PE. When $[K_1 : K_2] = [1 : 0], [1 : 1], [0 : 1]$, it is a Frobenius solution corresponding respectively to $t = t_1, t_2, t_3$, i.e., to $t = 0, 1, \infty$, in the following way: it belongs to the characteristic exponent μ'_i at the singular point $t = t_i$, and to the exponents μ_j, μ_k at the other two singular points. Of the five pseudo-Euclidean families, e.II(n) and e.IV($n, 1, \infty$) are integrated in [12, Tab. VIII], and the others are integrated similarly. \square

For several illustrative base gDH systems selected from Table 6, explicit integrations are supplied below. (See Examples 4.4–4.7.) Most come from the ODE's given in Table 7, via the $t(\cdot) \mapsto x(\cdot)$ map.

The simplest cases in Table 7 are the ones in which the curve in t, \dot{t} is rational, i.e. of genus 0, so that the solutions $t = t(\tau)$ can be expressed in terms of elementary functions. In the cases flagged as elliptic, the curve is of genus 1 and $t = t(\tau)$ can be expressed in terms of Weierstrass \wp -functions $\wp(g_2, g_3; \cdot)$. In most though not all of these cases the Klein–Weber invariant $j = g_2^3/(g_2^3 - 27g_3^2)$ of the curve is independent of K_1, K_2 , and equals either 0 (indicating an equianharmonic curve, and a triangular period lattice for $t(\cdot)$ and $x(\cdot)$); or 12^3 (indicating a lemniscatic curve, and a square period lattice). Incidentally, the first three curves in $\psi, d\psi/d\sigma$ appearing in Theorem 4.3 are elliptic, with respective j -values $0, 12^3, 0$; the final two are rational.

If an algebraic curve in t, \dot{t} is elliptic with $j = 0$ then $t(\cdot)$ and $x(\cdot)$ can optionally be expressed in terms of Dixon's elliptic functions sm, cm , which satisfy $\text{sm}^3 + \text{cm}^3 = 1$ and $(\text{sm})' = \text{cm}^2$, $(\text{cm})' = -\text{sm}^2$. (See [21]; sm, cm respectively equal $6\wp/(1 - 3\wp)$ and $(3\wp + 1)/(3\wp - 1)$, with $(g_2, g_3) = (0, 1/27)$ so that $\wp = 6\wp^2$.) If $j = 12^3$, they can be expressed in terms of the lemniscatic sine/cosine functions sl, cl , which satisfy $\text{sl}^2 + \text{cl}^2 = 1$ and $[(\text{sl})']^2 = 1 - \text{sl}^4$. These are identical to the Jacobi functions sn, cn that have modular parameter $m = k^2 = -1$. Jacobi functions with $m = k^2 = 1/2$ or $m = k^2 = 2$ could also be used.

It is noteworthy that besides expressing $t = t(\tau)$ and $x = x(\tau)$ in closed form, for any gDH system in Table 7 it is straightforward to construct first integrals that are rational in x_1, x_2, x_3 . It follows from $\dot{t}^{\bar{n}} = K_1 f^{(1)} + K_2 f^{(2)}$ that

$$(4.6) \quad I := \frac{\dot{f}^{(1)}(t) - \bar{n}(\ddot{t}/\dot{t})f^{(1)}(t)}{\dot{f}^{(2)}(t) - \bar{n}(\ddot{t}/\dot{t})f^{(2)}(t)} \Big|_{t=-(x_2-x_3)/(x_1-x_2)}$$

is a constant of the motion ($\dot{I} = 0$), i.e., is a first integral. (A useful expression for \ddot{t}/\dot{t} in terms of x_1, x_2, x_3 is provided by Eq. (2.40) of Lemma 2.6.) The same is true if $f^{(1)}, f^{(2)}$ are replaced by any two nonzero solutions of the PE, i.e., by any two nonzero combinations of $f^{(1)}, f^{(2)}$.

First integrals can also be constructed as products of powers of Darboux polynomials (sometimes called ‘second integrals’), if the space of such polynomials is sufficiently rich [27, 52]. By definition, a Darboux polynomial (DP) of a HQDS is a homogeneous polynomial $p \in \mathbb{C}[x_1, \dots, x_d]$ satisfying $\dot{p} := \sum_{i=1}^d Q_i \partial_i p = \lambda \cdot p$ for some ‘eigenvalue’ $\lambda \in \mathbb{C}[x_1, \dots, x_d]$, which if nonzero is necessarily a homogeneous polynomial of degree 1. For any DP p , the HQDS flow stabilizes the surface $p(x_1, \dots, x_d) = 0$. The DP's of any gDH system (1.4), with $d = 3$, include $x_2 - x_3$,

$x_3 - x_1$, $x_1 - x_2$, so the gDH flow stabilizes their zerosets, which are planes. (Note that by the $x(\cdot) \mapsto t(\cdot)$ map (2.34), these ‘planes of coincidence’ correspond to $t \equiv t_1, t_2, t_3$.) Also, if $c - a_i - b_i = 0$ for any i then x_i is a DP; i.e. the gDH flow also stabilizes the coordinate plane $x_i = 0$. If $c - a_1 - a_2 - a_3 = 0$, so that the system is improper, then any linear combination of $x_2 - x_3$, $x_3 - x_1$, $x_1 - x_2$ is a DP. That is, the flow stabilizes any plane containing the ray $(x_1, x_2, x_3) \propto (1, 1, 1)$; hence every solution $x = x(\tau)$ lies in an invariant plane. All these statements about DP’s can be rephrased in terms of proper subalgebras of the non-associative algebra \mathfrak{A} .

Example 4.4. gDH(1, 1, 0; 0, 0, 0; 1), i.e., the $n = 1$ system II, which has alternative parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}) = (-2, 0; -2, 0; 0, 1; 2)$ and has $(r_1, r'_1; r_2, r'_2; r_3, r'_3) = (1, \infty; 1, \infty; \infty, -2)$. Explicitly, this system is

$$(4.7) \quad \begin{cases} \dot{x}_1 = x_1(x_1 - x_2 - x_3), \\ \dot{x}_2 = x_2(x_2 - x_1 - x_3), \\ \dot{x}_3 = -x_1x_2. \end{cases}$$

This proper gDH system with the PP, invariant under $x_1 \leftrightarrow x_2$, is listed in Table 7. The algebraic curve in t, \dot{t} listed there,

$$(4.8) \quad \dot{t}^2 = K_1 t^2 - K_2 (2t - 1),$$

is rational. Up to the action of an affine transformation that replaces τ by $A\tau + B$, solving this ODE for $t = t(\tau)$ yields the following general solution of the gSE:

$$(4.9) \quad t(\tau) = \frac{c_1 + c_3 + c_2 \sin \tau}{2c_3},$$

where $[c_1 : c_2 : c_3]$ is any point on a genus-0 parametrization curve \mathfrak{C} , given in projective form (i.e., $\mathfrak{C} \subset \mathbb{P}^2$) as

$$(4.10) \quad \mathfrak{C}: \quad c_1^2 - c_2^2 - c_3^2 = 0.$$

Substituting (4.9) into the $t(\cdot) \mapsto x(\cdot)$ map (2.32) yields

$$(4.11) \quad x(\tau) = \left(\frac{-c_2 - (c_1 + c_3) \sin \tau}{(c_1 + c_3 + c_2 \sin \tau) \cos \tau}, \frac{-c_2 - (c_1 - c_3) \sin \tau}{(c_1 - c_3 + c_2 \sin \tau) \cos \tau}, -\tan \tau \right)$$

as the general solution of the gDH system, up to an affine transformation that replaces τ by $A\tau + B$ and scales x by A .

The points $[c_1 : c_2 : c_3] = [-1 : 0 : 1], [1 : 0 : 1], [\pm 1 : 1 : 0]$ on the curve \mathfrak{C} are special: for each, the solution (4.11) is coincident, because respectively $t \equiv 0, 1, \infty$, i.e., $t \equiv t_1, t_2, t_3$, which correspond to the planes $x_2 - x_3 = 0$, $x_3 - x_1 = 0$, $x_1 - x_2 = 0$. But these points on \mathfrak{C} can be shown by a limiting procedure to yield respective special solutions $t = t^{[i']}(\tau)$, $i = 1, 2, 3$, i.e.,

$$(4.12) \quad t^{[1']}(\tau) = 0 + e^\tau, \quad t^{[2']}(\tau) = 1 + e^\tau, \quad t^{[3']}(\tau) = 1/2 - \tau^2,$$

in each of which $\tau \mapsto A\tau + B$ can be taken. (As an alternative to the limiting procedure, in the ODE (4.8) for $t = t(\tau)$ simply set $[K_1 : K_2]$ equal to $[1 : 0], [1 : 1], [0, 1]$, corresponding to $t = 0, 1, \infty$.) In fact these special solutions of the gSE come from Frobenius solutions $f^{[1]}, f^{[2]}, f^{[3]}$ of the PE at $t = 0, 1, \infty$. They are of the type shown in (4.1b), associated respectively to the exponents $(r'_1, r'_2, r'_3) = (\infty, \infty, -2)$; and the first two have no singularities because that is what $r'_i = \infty$ implies. The general solution $t = t(\tau)$ given in (4.9) comes from deformations of

$f^{[1]}, f^{[2]}, f^{[3]}$. The (noncoincident) special gDH solutions $x = x^{[i]}(\tau)$, $i = 1, 2, 3$, obtained from the gSE solutions (4.12) by the $t(\cdot) \mapsto x(\cdot)$ map (2.32), are

$$(4.13a) \quad x = x^{[1]}(\tau) = (0, (1 - e^\tau)^{-1}, 1),$$

$$(4.13b) \quad x = x^{[2]}(\tau) = ((1 + e^\tau)^{-1}, 0, 1),$$

$$(4.13c) \quad x = x^{[3]}(\tau) = \left(\frac{1 + 2\tau^2}{\tau(1 - 2\tau^2)}, \frac{1 - 2\tau^2}{\tau(1 + 2\tau^2)}, \frac{1}{\tau} \right).$$

Again, an affine transformation can be applied. This completes the integration of the gDH system (4.7), insofar as noncoincident solutions are concerned. It should be noted that though the general solution (4.11) and the special solutions (4.13a), (4.13b) are simply periodic on the τ -plane, the curious 5-pole special solution (4.13c) is not.

Rational first integrals of this gDH system include

$$(4.14) \quad I_1 = x_3^2 - x_1 x_2, \quad I_2 = \frac{x_1(x_2 - x_3)^2}{x_1 - x_2}, \quad I_3 = \frac{x_2(x_3 - x_1)^2}{x_1 - x_2},$$

which satisfy $I_1 = I_2 - I_3$. In each, each factor in the numerator and denominator is a Darboux polynomial.

As is indicated in Table 6, the system $\text{gDH}(1, 2, 0; -1, 1, 0; 1)$, i.e., the $n = 1$ system II.1, is the image of this gDH system (the ‘base’) under $\sigma_{(12)} \circ \mathbf{2} \circ \sigma_{(23)}$. This is essentially the quadratic map given in (3.1a), with interchanges of components. As a solution-preserving map, it takes the above general and special solutions of the base system to those of the image, which like the base has the PP.

Example 4.5. $\text{gDH}(1, 1, 1; 0, 0, 0; 2)$, i.e., the $n = 1$ system V, which has alternative parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}) = (-2, 1; -2, 1; -2, 1; 2)$ and has $(r_1, r'_1; r_2, r'_2; r_3, r'_3) = (1, -2; 1, -2; 1, -2)$. Explicitly, this system is

$$(4.15) \quad \begin{cases} \dot{x}_1 = x_1^2 - (x_1 x_2 + x_2 x_3 + x_3 x_1), \\ \dot{x}_2 = x_2^2 - (x_1 x_2 + x_2 x_3 + x_3 x_1), \\ \dot{x}_3 = x_3^2 - (x_1 x_2 + x_2 x_3 + x_3 x_1). \end{cases}$$

This proper gDH system with the PP, invariant under arbitrary permutations of x_1, x_2, x_3 (an \mathfrak{S}_3 symmetry, cf. Theorem 3.4), is listed in Table 7. The algebraic curve in t, \dot{t} listed there, which is taken from [12], is generically elliptic and equianharmonic ($j = 0$). But it assumes that t_1, t_2, t_3 are $0, 1, \infty$. Because of the \mathfrak{S}_3 symmetry and in particular the $\mathfrak{3}_3$ (i.e., $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1$) symmetry, it is more convenient to choose t_1, t_2, t_3 to be $1, \omega, \omega^2$, where ω is a primitive cube root of unity. By examination, the curve in t, \dot{t} then becomes

$$(4.16) \quad \dot{t}^2 = K_1 (t - 1)^3 (t + 1) + K_2 (t - \omega)^3 (t + \omega) + K_3 (t - \omega^2)^3 (t + \omega^2),$$

in which for each i , K_i multiplies a Frobenius solution of the PE at $t = t_i$; so that one of K_1, K_2, K_3 is redundant, the PE solution space being 2-dimensional.

Up to the action of an affine transformation that replaces τ by $A\tau + B$, integrating this ODE yields the following general solution $t = t(\tau)$ of the gSE:

$$(4.17) \quad t(\tau) = \frac{\pm \sqrt{c_2 c_3^3} \dot{\varphi}(\tau) - 2 c_1 c_3 \varphi(\tau) + c_2^2}{4 c_3^2 \varphi^2(\tau) + c_1 c_2},$$

where $[c_1 : c_2 : c_3]$ is any point on a parametrization curve \mathfrak{C} , given in projective form (i.e., $\mathfrak{C} \subset \mathbb{P}^2$) as

$$(4.18) \quad \mathfrak{C}: \quad c_1^3 - c_2^3 - (4/27) c_3^3 = 0,$$

and the Weierstrass function \wp is equianharmonic, having parameters $(g_2, g_3) = (0, 1/27)$. (The curve \mathfrak{C} is elliptic, i.e. is of genus 1, and has $j = 0$, i.e., is itself equianharmonic.) The $t(\cdot) \mapsto x(\cdot)$ map (2.32) reduces for this gDH system to

$$(4.19) \quad x_i(\tau) = \frac{1}{2} \frac{d}{d\tau} \log \left[\frac{\dot{t}}{(t - \omega^{i-1})^2} \right] = \frac{\ddot{t}}{2\dot{t}} - \frac{\dot{t}}{t - \omega^{i-1}}, \quad i = 1, 2, 3,$$

and substituting (4.17) into (4.19) yields the general gDH solution $x = x(\tau)$, up to an affine transformation that replaces τ by $A\tau + B$ and scales x by A .

The points $[c_1 : c_2 : c_3] = [1 : 1 : 0], [1 : \omega : 0], [1 : \omega^2 : 0]$ on the curve \mathfrak{C} are special: for each, the resulting solution $x = x(\tau)$ is coincident, because respectively $t \equiv 1, \omega, \omega^2$, i.e., $t \equiv t_1, t_2, t_3$, which correspond to the planes $x_2 - x_3 = 0$, $x_3 - x_1 = 0$, $x_1 - x_2 = 0$. But these points on \mathfrak{C} can be shown by a limiting procedure to yield respective special solutions $t = t^{[i']}(\tau)$, $i = 1, 2, 3$, i.e.,

$$(4.20) \quad t^{[i']}(\tau) = \omega^{i-1} \frac{\tau^2 + 1}{\tau^2 - 1}, \quad i = 1, 2, 3,$$

in each of which $\tau \mapsto A\tau + B$ can be taken. (As an alternative to the limiting procedure, in the ODE (4.16) for $t = t(\tau)$ simply set $[K_1 : K_2 : K_3]$ equal to $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$, corresponding to $t = 1, \omega, \omega^2$.) In fact these special solutions of the gSE come from Frobenius solutions $f^{[1]}, f^{[2]}, f^{[3]}$ of the PE at $t = 1, \omega, \omega^2$. They are of the type shown in (4.1b), associated respectively to the exponents $(r'_1, r'_2, r'_3) = (-2, -2, -2)$; and the exponents determine their behavior as $\tau \rightarrow \infty$. The (noncoincident) special gDH solutions $x = x^{[i']}(\tau)$ obtained from these gSE solutions by the $t(\cdot) \mapsto x(\cdot)$ map (4.19) include

$$(4.21) \quad x = x^{[1']}(\tau) = \left(\frac{1}{2\tau}, \frac{1 - 3\tau^2}{2\tau(1 + \tau^2)}, \frac{1 + 3\tau^2}{2\tau(1 - \tau^2)} \right),$$

with $x^{[2]}, x^{[3]}$ obtained from $x^{[1']}$ by cyclically permuting components. Again, an affine transformation can be applied. This completes the integration of the gDH system (4.15), insofar as noncoincident solutions are concerned. It should be noted that though the general solution is doubly periodic in τ , the three special solutions, such as (4.21), are rational (with 5 poles) and are not periodic.

Rational first integrals of this gDH system include I_1, I_2, I_3 , defined as

$$(4.22) \quad I_i = \frac{(x_i - x_j)^3 P_k(x_1, x_2, x_3)}{(x_j - x_k)^3 P_i(x_1, x_2, x_3)},$$

where i, j, k are a cyclic permutation of 1, 2, 3. In this, for $i = 1, 2, 3$, the polynomial $P_i(x_1, x_2, x_3)$ is defined to be $3x_i^2 - x_1x_2 - x_2x_3 - x_3x_1$. Each of P_1, P_2, P_3 is a Darboux polynomial, like $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$, and the degree-6 polynomial $P_1P_2P_3$ is not merely a Darboux polynomial: it is a first integral.

As mentioned, and as is indicated in Table 6, the $n = 1$ systems V.1, V.2, V.3 are the images of this gDH system (V) under the rational morphisms **2**, **6_c**, **3_c**. (Some interchanges of components are required.) The most interesting of these maps is the $V \mapsto V.2$ one, performed by $\sigma_{(13)} \circ \mathbf{6}_c$, because it will be shown in Part II that the third component of the $n = 1$ system V.2, i.e., of gDH(3, 2, 3; -2, -2, 4; 2),

satisfies the $N = 4$ case of the Chazy-XI equation. The Chazy-XI equation is a third-order scalar ODE with the PP, which is of the Chazy-class form (1.3) with $[A : B : C : D] \in \mathbb{P}^1(1, 1, 2, 3)$ parametrized by a positive integer N and equal to

$$(4.23) \quad [N^2 - 1 : N^2 - 13 : 12(N^2 - 1) : -3(N^2 - 1)^2].$$

(See [17, 59]; for the PP one needs $N \neq 1$ and $6 \nmid N$.) Explicitly, it is

$$(4.24) \quad \ddot{u} = \lambda(N^2 - 1) u \ddot{u} + \lambda(N^2 - 13) \dot{u}^2 + 12\lambda^2(N^2 - 1) u^2 \dot{u} - 3\lambda^3(N^2 - 1)^2 u^4,$$

where the value of λ is a matter of convention; it simply scales u . For the $N = 4$ case arising from this gDH system, it happens that $\lambda = 2/5$ is appropriate.

According to the formula $x = \Phi(\tilde{x})$ for the $\mathbf{6_c}$ map given in (3.35), the third component of the image system under $\sigma_{(13)} \circ \mathbf{6_c}$ is simply $(\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)/3$. Hence if $x = x(\tau)$ is any of the above solutions of the $n = 1$ system V, general or special, then the average of components $u := (x_1 + x_2 + x_3)/3$ will be a solution of Chazy-XI(4), i.e., of Eq. (4.24) with $N = 4$ and $\lambda = 2/5$.

Thus the general solution $x = x(\tau)$ of the gDH system (4.15), computed from (4.17) by (4.19), yields the general solution $u = u(\tau)$ of Chazy-XI(4). Explicitly,

$$(4.25) \quad u(\tau) = \frac{1}{2} \frac{d}{d\tau} \log \left[\frac{\dot{t}}{(t^3 - 1)^{2/3}} \right] = \frac{\ddot{t}}{2\dot{t}} - \frac{t^2 \dot{t}}{t^3 - 1},$$

where $t = t(\tau)$ is the general gSE solution (4.17). This general solution is rational in the equianharmonic Weierstrass functions \wp, ϕ , and is therefore doubly periodic with a triangular period lattice; and it is algebraically parametrized by the point $[c_1 : c_2 : c_3] \in \mathfrak{C} \subset \mathbb{P}^2$. Similarly, each of the special gDH solutions $x = x^{[i']}(\tau)$, such as $x^{[1']}(\tau)$ of (4.21), yields the special (i.e. rational) Chazy-XI(4) solution

$$(4.26) \quad u = u^{[1']}(\tau) = u^{[2']}(\tau) = u^{[3']}(\tau) = \frac{1 - 5\tau^4}{2\tau(1 + 3\tau^4)},$$

up to an affine transformation. This too solves (4.24) with $N = 4$ and $\lambda = 2/5$.

Any Chazy solution $u = u(\tau)$ is transformed to another Chazy solution by any affine transformation that replaces τ by $A\tau + B$ and scales u by A . It should be mentioned that there are additional special Chazy-XI(4) solutions,

$$(4.27) \quad u(\tau) = \frac{1}{2\tau}, \quad -\frac{5}{6\tau}, \quad -\frac{1}{3\tau},$$

which fit less well into the framework of PE-based integration. These come via the averaging formula $u = (x_1 + x_2 + x_3)/3$ from ray solutions $x = x(\tau)$ of the gDH system (4.15), each of which has coincident components. As is easily verified, the respective ray solutions are the ones along the directions e_0, e'_i , and e_i . (See Proposition 2.3; by the \mathfrak{S}_3 symmetry, i is arbitrary.)

It is not difficult to generalize heuristically the special Chazy-XI(4) solutions (4.26), (4.27) to the case of arbitrary N (and trivially, to arbitrary λ). The generalizations, which solve Eq. (4.24) and subsume (4.26), (4.27), are

$$(4.28) \quad \lambda u(\tau) = \frac{c_+ \tau^{N/2} + c_- \tau^{-N/2}}{\tau [(1 - N)c_+ \tau^{N/2} + (1 + N)c_- \tau^{-N/2}]}, \quad \frac{2}{(1 - N^2)\tau},$$

in the first of which $c_+, c_- \in \mathbb{C}$ are parameters satisfying $c_+ + c_- = 1$; equivalently, $[c_+ : c_-] \in \mathbb{P}^1$. To both, a translation $\tau \mapsto \tau + B$ can be applied. The above special solutions (with $N = 4$) are recovered by setting $[c_+ : c_-] = [-5 : 1], [0 : 1], [1 : 0]$.

The *general* solution (4.25) of Chazy-XI(4), rational in the equianharmonic Weierstrass functions \wp, \wp' , cannot readily be generalized from $N = 4$ to arbitrary N ; not even to the integral values of N for which Chazy-XI has the Painlevé property. But it will be shown in Part II that the individual components $x_i = x_i(\tau)$, $i = 1, 2, 3$, of the gDH system (4.15) satisfy Chazy-XI(2), with $\lambda = 2/3$. Also, it will be shown that the first component of the system gDH(4, 6, 4; 3, -1, -1; 2), i.e., of the $n = 2$ system III.1 listed in Table 6, satisfies Chazy-XI(9), with $\lambda = 1/10$. So the general solution of Chazy-XI(N) with $N = 2, 4, 9$, at least, can be constructed by PE-based integration, applied to a proper gDH system with the PP. Chazy-XI(N) is integrable in a Liouvillian sense for any N [17], but constructing algebraically parametrized closed-form solutions may be especially easy for certain integer values of N .

Example 4.6. The parametrized gDH family e.IV(n, q, r), defined by either of

(4.29a)

$$(a_1, a_2, a_3; b_1, b_2, b_3; c) \propto (n+1, r(n+1), qn - n - 1; qn - n - 1, -q, n+1; qn),$$

$$(4.29b) \quad (\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3) = \left(-\frac{\bar{n}}{r}, \frac{q-\bar{n}}{r}; -\bar{n}, -\bar{n}+1; -\frac{q-\bar{n}}{r}, \frac{\bar{n}}{r}\right),$$

where as usual $\bar{n} := (n+1)/n$, with n restricted by $n \in \mathbb{Z} \setminus \{0, -1, -2\}$; and where q is a positive integer, and r a nonzero one. (By assumption $n \neq \infty$ here, so that $\bar{n} \neq 1$; the family e.IV(∞, q, r) is best treated separately.)

One notes that the second angular parameter, $\alpha_2 := \nu'_2 - \nu_2$, always equals 1. Also, from the formula $(r_i, r'_i) = -\bar{n}(1/\nu_i, 1/\nu'_i)$, it follows that the vector of leading exponents of the Frobenius-derived gSE solutions shown in (4.1) is

$$(4.30) \quad (r_1, r'_1; r_2, r'_2; r_3, r'_3) = \left(r, \frac{-(n+1)r}{qn-n-1}; 1, n+1; \frac{(n+1)r}{qn-n-1}, -r\right),$$

in which the pair $(r_2, r'_2) = (1, n+1)$ is what one would expect of an *ordinary*, nonsingular point. In fact, in the gSE associated to any gDH system in the family e.IV(n, q, r), the second ‘singular point’ $t = t_2$ is not a singular point at all: it is ordinary. This is because, according to (4.29b),

$$(4.31) \quad (\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3) = (\nu, \nu'; -\bar{n}, -\bar{n}+1; -\nu', -\nu)$$

for certain ν, ν' . If the vector of offset exponents is of this special form, then no poles will appear at $t = t_2$ in the gSE (2.41), as one can easily check.

A related degeneracy is visible in the gDH parameter vector (4.29a). It satisfies $c - a_1 - b_1 = 0$ and $c - a_3 - b_3 = 0$. From this, it follows that the gDH flow stabilizes the coordinate planes $x_1 = 0$ and $x_3 = 0$. Moreover, any system e.IV(n, q, r) is ‘triangular’: x_1, x_3 evolve independently of x_2 . Although the opposite is not true, the non-associative algebra \mathfrak{A} associated to the system does have as a proper subalgebra the span of e_2 .

The systems in this family that have the PP are listed in both Tables 6 and 7. For e.IV(n, q, r) and its associated gSE to have the PP, (n, q) must be tightly restricted, because of the following considerations. The auxiliary exponents q_i, q'_i (see (4.3)) are defined by $(q_i, q'_i) = \bar{n}((\nu_i - \nu'_i)/\nu_i, (\nu'_i - \nu_i)/\nu'_i)$, so that

$$(4.32) \quad (n+1)^{-1} + q_i^{-1} + q'_i{}^{-1} = 1, \quad i = 1, 2, 3,$$

as was previously noted. Moreover, one calculates from (4.29b) that

$$(4.33) \quad (q_1, q'_1; q_2, q'_2; q_3, q'_3) = (q, q'; 1, -(n+1); q', q)$$

for a certain q' (namely, $q' = \frac{(n+1)q}{qn-n-1}$.) Equation (4.32) thus requires that $n+1$ (an integer other than $1, 0, -1$), q (an integer), and q' (an integer or ∞) be related by the sum of their reciprocals equaling 1. Hence, $\{n+1, q, q'\}$ must be one of $\{n+1, 1, -(n+1)\}$, $\{2, 2, \infty\}$, $\{2, 3, 6\}$, $\{2, 4, 4\}$, or $\{3, 3, 3\}$; so that

$$(4.34) \quad (n, q) = (n, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 2), (5, 2),$$

up to the interchange $q \leftrightarrow q'$, or equivalently up to $x_1 \leftrightarrow x_3$. These choices yield the systems $\text{e.IV}(n, q, r)$ with the PP that appear in Tables 6 and 7. It should be noted that the final choice $(n, q) = (5, 2)$ is anomalous. As is indicated in Table 6, the system $\text{e.IV}(5, 2, r)$ will have the PP only if r is even.

For any system $\text{e.IV}(n, q, r)$, having the PP or not, it follows readily from PE-based integration that the algebraic curve in t, \dot{t} that determines the gSE and gDH solutions (i.e., the ODE satisfied by $t = t(\tau)$) is

$$(4.35) \quad \dot{u}^{n+1} = (K_1 u^q - K_2)^n, \quad t := u^r,$$

as is indicated in Table 7. This is a statement that the curve in t, \dot{t} is the image under a cyclic covering $t = R(u) := u^r$ of a curve in u, \dot{u} ; and associated to this covering there must necessarily be a rational morphism (i.e. solution-preserving map) $x = \Phi(\tilde{x})$, from $\text{e.IV}(n, q, 1)$ to $\text{e.IV}(n, q, r)$. The latter system is therefore an image system, and its integration is facilitated by the existence of the map. It follows from the useful formula (3.10) that the map is

$$(4.36) \quad x_1 = \tilde{x}_1, \quad x_2 = \frac{\tilde{x}_1(\tilde{x}_3 - \tilde{x}_2)^r - \tilde{x}_3(\tilde{x}_1 - \tilde{x}_2)^r}{(\tilde{x}_3 - \tilde{x}_2)^r - (\tilde{x}_1 - \tilde{x}_2)^r}, \quad x_3 = \tilde{x}_3,$$

which is of a new type. It did not arise in §3 because the PE associated to any $\text{e.IV}(n, q, r)$, having only two singular points on \mathbb{P}_t^1 as explained above (i.e., t_1, t_3 but not t_2), is of a degenerate type that was not considered there.

Written explicitly, even the base system $\text{e.IV}(n, q, 1)$ is disconcertingly complicated:

$$(4.37) \quad \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_1[(n+1)\tilde{x}_1 - (n+q+1)\tilde{x}_3], \\ \dot{\tilde{x}}_2 = (n+1)\tilde{x}_2^2 + \tilde{x}_3[qn - 2n - 2]\tilde{x}_2 - (q-1)(n+1)\tilde{x}_1], \\ \dot{\tilde{x}}_3 = \tilde{x}_3[qn - n - 1]\tilde{x}_3 - (q-1)(n+1)\tilde{x}_1]. \end{cases}$$

But the algebraic curve in u, \dot{u} (i.e., $\tilde{t}, \dot{\tilde{t}}$) used in its integration will simply be the curve (4.35), which is easy to integrate. Once the gSE associated to the base system $\text{e.IV}(n, q, 1)$ has been integrated, i.e., its general and special solutions $t = t(\tau)$ have been found, the gDH solutions $\tilde{x} = \tilde{x}(\tau)$ are obtained by applying the $t(\cdot) \mapsto x(\cdot)$ map. From these, explicit solutions of the image system $\text{e.IV}(n, q, r)$, for r any nonzero integer, can be obtained by applying the solution-preserving map (4.36).

As Table 7 indicates, the curve (4.35) in u, \dot{u} is rational for (n, q) equal to $(n, 1)$ or $(1, 2)$; generically elliptic (with $j = 0$) for (n, q) equal to $(1, 3)$, $(2, 2)$, or $(2, 3)$; and generically elliptic (with $j = 12^3$) for (n, q) equal to $(1, 4)$ or $(3, 2)$. In the rational cases the construction of explicit gSE solutions $t = t(\tau)$ proceeds similarly to Example 4.4; and in the elliptic cases, to Example 4.5. (Details are left to the reader.) Only the case $(n, q) = (5, 2)$ requires special comment. As is indicated in Table 7, for this choice the curve (4.35) is generally hyperelliptic. Its general integral $u = u(\tau)$ is not single-valued on the complex τ -plane; but if r is even then $t := u^r$ will be single-valued. In fact, it will be an equianharmonic elliptic function

of τ . Another way of saying this is that if $v := u^2$ then the resulting curve in v, \dot{v} , which is

$$(4.38) \quad \dot{v}^6 = 64 v^3 (K_1 v - K_2)^5,$$

is elliptic (with $j = 0$), provided that $K_1 K_2 \neq 0$. This completes our summary of the integration of the cases (4.34) of the gDH system (4.37) that have the PP, insofar as noncoincident solutions are concerned.

Rational first integrals of the base gDH system e.IV($n, q, 1$) include

$$(4.39) \quad I_1 = \tilde{x}_1^{(q-1)n-1} \tilde{x}_3^{n+1} (\tilde{x}_3 - \tilde{x}_1)^q, \quad I_2 = \frac{\tilde{x}_1 (\tilde{x}_3 - \tilde{x}_2)^q}{\tilde{x}_3 (\tilde{x}_1 - \tilde{x}_2)^q}.$$

In each, each factor is a Darboux polynomial. For any gDH system e.IV(n, q, r) with r a nonzero integer other than ± 1 , the corresponding first integrals are algebraic rather than rational. They can be derived from (4.39) by applying a irrational but algebraic morphism that is the inverse of the morphism (4.36).

It must be stressed that I_1, I_2 of (4.39) are first integrals of the system (4.37), i.e., of

$$\text{e.IV}(n, q, 1) = \text{gDH}(n+1, n+1, qn-n-1; qn-n-1, -q, n+1; qn),$$

whether or not it appears in Table 7, i.e., whether or not the pair (n, q) takes on one of the relatively few values that cause it to have the Painlevé property. The parameters n, q do not even need to take on integral values.

Example 4.7. gDH(0, 0, 0; 0, 1, 0; 1), i.e., the $n = \infty$ system e.IV($\infty, 1, \infty$), which has alternative parameter vector $(\nu_1, \nu'_1; \nu_2, \nu'_2; \nu_3, \nu'_3; \bar{n}) = (0, 0; 0, -1; 0, 0; 1)$ and has $(r_1, r'_1; r_2, r'_2; r_3, r'_3) = (\infty, \infty; \infty, 1; \infty, \infty)$. Explicitly, this system is

$$(4.40) \quad \begin{cases} \dot{x}_1 = x_3(x_1 - x_2), \\ \dot{x}_2 = 0, \\ \dot{x}_3 = x_1(x_3 - x_2). \end{cases}$$

This proper but rather degenerate gDH system with the PP, invariant under $x_1 \leftrightarrow x_3$, does not appear in Table 7: it is the $n = \infty$ member of e.IV($\infty, 1, n$), the fifth pseudo-Euclidean family treated in Theorem 4.1. The curve in t, \dot{t} is not algebraic but transcendental. It was not given in the theorem but is

$$(4.41) \quad \dot{t} = K_1(t \log t) + K_2 t,$$

the right side being the solution $K_1 f^{(1)}(t) + K_2 f^{(2)}(t)$ of the PE. As in Example 4.6 above, the second ‘singular point’ $t = t_2$ of the PE (i.e., $t = 1$) is ordinary rather than singular: the only singular points are $t = t_1, t_3$, i.e., $t = 0, \infty$.

Up to an affine transformation that replaces τ by $A\tau + B$, the general solution of (4.41) is

$$(4.42) \quad t(\tau) = \exp(C + e^\tau) =: \bar{C} \exp(e^\tau),$$

as stated in the theorem. Substituting the gSE solution (4.42) into the $t(\cdot) \mapsto x(\cdot)$ map (2.32) yields

$$(4.43) \quad x = x(\tau) = \left(\frac{1 + e^\tau - \bar{C} \exp(e^\tau)}{1 - \bar{C} \exp(e^\tau)}, 1, \frac{1 - \bar{C}(1 - e^\tau) \exp(e^\tau)}{1 - \bar{C} \exp(e^\tau)} \right)$$

as the general solution of the gDH system, up to an affine transformation that replaces τ by $A\tau + B$ and scales x by A . Here, the free parameter $\bar{C} \in \mathbb{C}$ can be viewed as a point on a genus-0 parametrization curve $\mathfrak{C} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

The points $\bar{C} = 0, \infty$ on \mathfrak{C} are special: for each, the solution (4.43) is coincident, because respectively $t \equiv 0, \infty$, i.e., $t \equiv t_1, t_3$, which correspond to the planes $x_2 - x_3 = 0$, $x_1 - x_2 = 0$. But these points on \mathfrak{C} can be shown by a limiting argument to yield special gSE solutions

$$(4.44) \quad t^{[1']}(\tau) = \exp(e^\tau), \quad t^{[3']}(\tau) = e^\tau,$$

in each of which $\tau \mapsto A\tau + B$ can be taken. (Alternatively, simply set $[K_1 : K_2] = [0 : 1], [1 : 0]$ in (4.41).) The first of these is merely an instance of the general solution. But the solution $t = t^{[3']}(\tau) = e^\tau$, which was mentioned in Theorem 4.3, comes from a Frobenius solution $f^{[3']}$ of the PE at $t = t_3 = \infty$; and it is of the type shown in (4.1b), though it has no singularity because $r'_3 = \infty$. It is deformations of $f^{[3']}$ that yield the above general solution, $t(\tau) = \bar{C} \exp(e^\tau)$. The special gDH solution obtained from $t = t^{[3']}(\tau)$ by the $t(\cdot) \mapsto x(\cdot)$ map is

$$(4.45) \quad x = x^{[3']}(\tau) = \left(\frac{1}{1 - e^\tau}, 0, \frac{e^\tau}{1 - e^\tau} \right).$$

Again, an affine transformation can be applied (e.g., the one based on a replacement of τ by $-\tau$ will interchange x_1, x_3). This completes the integration of the gDH system (4.40), insofar as noncoincident solutions are concerned.

Because of the double exponentials in (4.43), the poles of the general solution $x = x(\tau)$ in the complex plane form a lattice that is not regular, being exponentially stretched. Such stretching, polynomial for n finite and exponential for $n = \infty$, is characteristic of gDH systems in the pseudo-Euclidean families e.I.1(n), e.I.2(n), e.II(n), e.IV.4(n), and e.IV($n, 1, \infty$), as Theorem 4.3 makes clear.

First integrals of this gDH system include I_1, I_2 , defined as

$$(4.46) \quad I_1 = x_2 \log \left(\frac{x_3 - x_2}{x_1 - x_2} \right) + (x_3 - x_1), \quad I_2 = x_2.$$

The transcendentality is apparent.

In each of the preceding examples of explicit (Papperitz-based) integration of non-DH gDH systems with the PP, there was a *general* noncoincident solution $x = x(\tau)$ with three free parameters, and one or more *special* noncoincident solutions having only two. (For both types of solution, one must include in the count the parameters $(A, B) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ of the affine transformation $\tau \mapsto A\tau + B$, which can always be applied.) In the integration of differential equations and systems, solutions without a full complement of free parameters are traditionally called ‘singular’ solutions [19]. This is what the above special solutions are.

Generically, special gDH and gSE solutions originate in the following way. Suppose that $\nu_i < \nu'_i$ for any of $i = 1, 2, 3$. There will be a gDH solution $x = x^{[i']}(\tau)$, derived from a gSE solution $t = t^{[i']}(\tau)$ of the type shown in (4.1b), with leading exponent r'_i ; the gSE solution comes from a Frobenius solution of the PE at the singular point $t = t_i$, with leading exponent μ'_i . There will also be a gDH solution $x = x^{[i]}(\tau)$, derived from a gSE solution $t = t^{[i]}(\tau)$ of the type shown in (4.1a), with leading exponent r_i ; the gSE solution comes from a Frobenius solution of the PE at $t = t_i$, with exponent μ_i . But $t = t^{[i']}(\tau)$ will be *special*, with only two free parameters, and $t = t^{[i]}(\tau)$ will lie on the three-dimensional general-solution manifold. This is because the dominance assumption $\nu_i < \nu'_i$ ensures that each instance of the general solution will have leading exponent r_i , not r'_i . By definition, each such

instance is based on a nontrivial mixture of the two Frobenius solutions at $t = t_i$; and it will be of the type shown in (4.3a), with a leading exponent (r_i) different from that of the special solution (r'_i) , which is ‘recessive.’

However, special solutions can be obtained from the general solution by a limiting procedure, as has been mentioned. It must also be mentioned that solutions with coincident components of any gDH system are ‘singular’ in the traditional sense; but they do not arise from PE-based integration of the sort considered here.

We close by briefly discussing possible extensions of the preceding classification of proper gDH systems with the PP. Any solution $x = x(\tau)$ of such a system is meromorphic on its maximal domain of definition in \mathbb{C} (which in the $\text{DH}(\frac{1}{N_1}, \frac{1}{N_2}, \frac{1}{N_3})$ case is generically a subdomain: a disk or half-plane). An extension would be a classification of (proper) gDH systems having the *weak* Painlevé property (wPP). Solutions of such systems are allowed to be finitely branched, i.e., to be meromorphic on a finite cover of a domain in \mathbb{C} . The classification of proper gDH systems with the wPP will presumably require a classification of gSE’s with the wPP, going beyond the work of Garnier and Carton-LeBrun. Such gSE’s may be numerous.

Another extension would be a fuller study of the integrability properties of gDH systems. Under what circumstances are there two (algebraically independent) first integrals, say rational or algebraic in x_1, x_2, x_3 ? The integrability properties of HQDS’s of Lotka–Volterra type have been exhaustively investigated, as was mentioned in the Introduction. But the class of gDH systems, which is nearly disjoint, remains to be treated. It is clear that on account of Papperitz-based integration, proper gDH systems may be not only Liouvillian-integrable, but also integrable in the preceding sense; even if they lack the PP. (See, e.g., the remarks at the end of Example 4.6 above.)

As a further example, let the parameters $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ equal $(a, a, a; b, b, b; c)$. The resulting \mathfrak{S}_3 -invariant gDH system can be denoted by $\text{gDH}(a; b; c)$. If $c - a - b = 0$, the gDH flow stabilizes each coordinate plane $x_i = 0$, and the first integrals include

$$(4.47) \quad I_i = \frac{x_i(x_j - x_k)}{x_j(x_k - x_i)}, \quad i = 1, 2, 3,$$

where i, j, k is a cyclic permutation of 1, 2, 3. (These satisfy $I_1 I_2 I_3 = 1$.) In each of these, each factor is a Darboux polynomial. The integrability of $\text{gDH}(a; b; c)$ when $c - a - b = 0$ has been noticed elsewhere [42, Ex. 2.5]. The special case $\text{gDH}(0; 1; 1)$ is a Lotka–Volterra model that has a bi-Hamiltonian structure, and its integration can be carried out in a manner that respects the structure [29, 30].

This paper has said little about *improper* gDH systems, because they are not amenable to Papperitz-based integration; and because if the parameter vector $(a_1, a_2, a_3; b_1, b_2, b_3; c)$ is not proper, the $\text{gDH} \leftrightarrow \text{gSE}$ correspondence of Theorem 2.9 breaks down. However, many improper gDH systems are rationally integrable. If $c - a_1 - a_2 - a_3 = 0$ then the system $\text{gDH}(a_1, a_2, a_3; b_1, b_2, b_3; c)$, improper by definition, stabilizes any plane containing the ray $(x_1, x_2, x_3) \propto (1, 1, 1)$, and has first integrals

$$(4.48) \quad I_i = \frac{x_i - x_j}{x_k - x_i}, \quad i = 1, 2, 3.$$

This makes possible the integration of any $\text{DH}(a_1, a_2, a_3; c)$ system that is improper on account of $c - a_1 - a_2 - a_3$ equaling zero [45, §4.3], even though no

proper $\text{DH}(a_1, a_2, a_3; c)$, i.e. no system $\text{DH}(\alpha_1, \alpha_2, \alpha_3 | c)$, is even algebraically integrable [47, 63]. This fact also lies behind several integrations of improper non-DH gDH systems that have appeared in the literature. One is the Kasner system $\text{gDH}(1, 1, 1; 1, 1, 1; 3)$, which is of historic interest. (See [40, 42] and [66, § 5.3].)

Any complete Painlevé analysis of improper gDH systems must examine all systems that are classified as improper because $c = 0$ or $2c - b_1 - b_2 - b_3 = 0$, as well as those with $c - a_1 - a_2 - a_3 = 0$. Owing to the absence of the $\text{gDH} \leftrightarrow \text{gSE}$ correspondence, a classification of the improper gDH systems that have the PP will not be attempted here.

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$$\begin{aligned}\frac{dx_1}{dt} &= a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2, \\ \frac{dx_2}{dt} &= b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2, \\ \frac{dx_3}{dt} &= c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2\end{aligned}$$

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